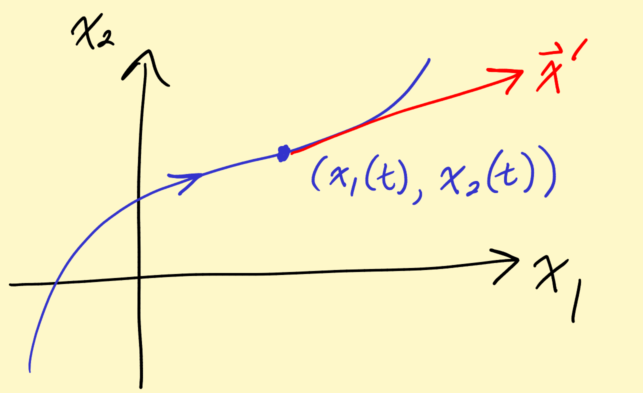
Lecture 24 Systems & linear algebra (5.1) MyLab HW 23

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 - 3x_2 & + f_1(t) \\[2ex] \dfrac{dx_2}{dt} = -2x_1 + 2x_2 & + f_2(t) \end{cases}$$

Homog first order linear system

non-homogeneous 1st order linear system

$$\vec{x} = \vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$



$$\vec{x}\,'(t) = \lim_{h\to 0} \frac{\vec{x}(t+h) - \vec{x}(t)}{h}$$

$$= \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}$$

$$\vec{x}\,' = \mathbb{A}\,\vec{x} \qquad \mathbb{A} = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$$

Try $\underline{\vec{x} = \vec{a}\, e^{rt}}$. Want $(\vec{a} e^{rt})' = \mathbb{A}\,\vec{a} e^{rt}$

$$r\vec{a} e^{rt} = \left[\mathbb{A}\vec{a}\right] e^{rt}$$

Need $\boxed{r\vec{a} = \mathbb{A}\,\vec{a}}$ ← eigenvector problem

$$\mathbb{A}\vec{a} - r\vec{a} = \vec{0}$$

$$(\mathbb{A} - r\mathbb{I})\vec{a} = \vec{0} \leftarrow \text{to get non-zero vector sol}^{\text{n}}\text{s } \vec{a}\text{, need}$$

$$\boxed{\det(\mathbb{A} - r\mathbb{I}) = 0} \leftarrow \text{Characteristic eqn}$$

$$\det\begin{bmatrix} 1-r & -3 \\ -2 & 2-r \end{bmatrix} = 0$$

$$(1-r)(2-r) - 6 = 0$$

$$r^2 - 3r - 4 = 0$$

$$(r-4)(r+1) = 0$$

Eigenvalues (e-vals) $r_1 = -1$, $r_2 = 4$.

Step 2 Find corresponding eigenvectors $\vec{a}$ (e-vects).

For $r = -1$: $(\mathbb{A} - r\mathbb{I})\vec{a} = \vec{0}$

$r = -1$

$$\begin{bmatrix} 1-(-1) & -3 \\ -2 & 2-(-1) \end{bmatrix} \vec{a} = \vec{0}$$

$$\left[\begin{array}{cc|c} 2 & -3 & 0 \\ -2 & 3 & 0 \end{array}\right] \quad \text{eqn } 2 = -(\text{eqn } 1)\,!$$

$$\sim\!\!\!\to \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad R_2 + R_1$$

Nice trick: Looking for non-zero vector $\vec{a}$.

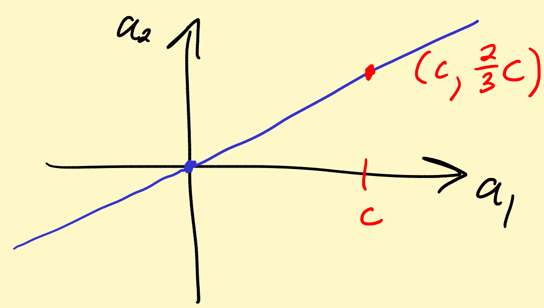
See one! $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$$\left[\begin{array}{cc|c} A & B & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \text{See } \begin{pmatrix} B \\ -A \end{pmatrix} \text{ or } \begin{pmatrix} -B \\ A \end{pmatrix}$$

$$\text{Get } \vec{x}_1 = \vec{a}\, e^{rt} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t} = \begin{pmatrix} 3e^{-t} \\ 2e^{-t} \end{pmatrix} \begin{array}{l} \leftarrow x_1(t) \\ \leftarrow x_2(t) \end{array}$$

or $\quad 2a_1 - 3a_2 = 0$

$$\boxed{a_2 = \tfrac{2}{3} a_1}$$



Get $\begin{pmatrix} c \\ \frac{2}{3}c \end{pmatrix}$. Pick a c to make it look nice.

I like $c = 3$ because it clears fractions. $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$c = 1$ is nice too. $\begin{pmatrix} 1 \\ 2/3 \end{pmatrix}$

For $r = 4$

$$\left[\begin{array}{cc|c} 1-4 & -3 & 0 \\ -2 & 2-4 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} -3 & -3 & 0 \\ \underline{-2} & \underline{-2} & 0 \end{array}\right] \leftarrow \text{expect this}$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ \underline{0} & \underline{0} & 0 \end{array}\right]$$

$A=1$, $B=1$ $\qquad \begin{pmatrix} B \\ -A \end{pmatrix}$ or $\begin{pmatrix} -B \\ A \end{pmatrix}$

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

like this one

$$\vec{x}_2 = \vec{a}\, e^{rt} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} = \begin{pmatrix} e^{4t} \\ -e^{4t} \end{pmatrix}$$

Superposition principle: $\vec{x} = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}$

is also a sol^n.

Is it the gen^l sol^n? Can I solve _any_ and _all_ IVP's : $\vec{x}(t_0) = \begin{pmatrix} A \\ B \end{pmatrix}$

Hmmm. Try $\vec{x}(t_0) = c_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t_0} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t_0} \overset{\text{want}}{=} \begin{pmatrix} A \\ B \end{pmatrix}$

$$\begin{bmatrix} 3e^{-t_0} & e^{4t_0} \\ 2e^{-t_0} & -e^{4t_0} \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

To be able to get c's no matter what $\binom{A}{B}$, need

$$\det \begin{bmatrix} 3e^{-t_0} & e^{4t_0} \\ 2e^{-t_0} & -e^{4t_0} \end{bmatrix} \neq 0$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \vec{x}_1(t_0) & \vec{x}_2(t_0) \end{array}$$

$$-3e^{3t_0} - 2e^{3t_0} = -5e^{3t_0} \leftarrow \text{not zero!}$$

$\uparrow$ never zero

We have gen^l sol^n.

<u>Def^n</u> Wronskian det. is $\det\left[\vec{x}_1(t), \vec{x}_2(t)\right] = W(t)$

<u>Abel's thm</u> If all the $p_{ij}(t)$'s in the linear system are continuous, then $W(t)$ is either

A) $\equiv 0$, or

B) never zero

on interval where all p's are continuous.

Consequently, can pick $t_0 = 0$ to test $W(t) \neq 0$.

<u>Meaning of $W(t_0) = 0$</u>   Take $t_0 = 0$.

$$\underbrace{\begin{bmatrix} \vec{x}_1 \, , & \vec{x}_2 \end{bmatrix}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If det $=0$, there exist non-zero vector sol^{n}s $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ to this system.

Suppose $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \neq \vec{0}$. Then $\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$ solves system with $\vec{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Zero sol^n does too!

$\overline{\text{U}}$. of E & U means $c_1 \vec{x}_1 + c_2 \vec{x}_2 \equiv \vec{0}$

One of $\vec{x}_1$ and $\vec{x}_2$ is a multiple of the other.

They are <u>linearly dependent</u>.

Basic problem 2x2 homog linear system:

Find <u>two</u> sol^{n}s $\vec{x}_1$ & $\vec{x}_2$ so that

A) $W[\vec{x}_1, \vec{x}_2](t) \neq 0$

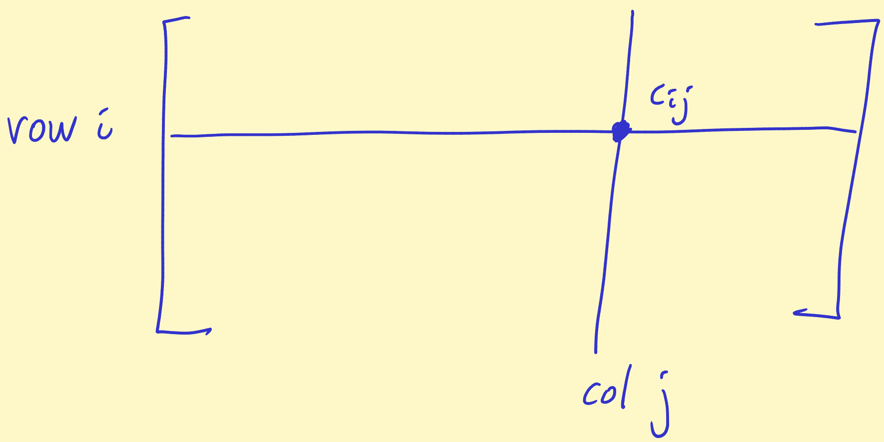
or <u>equivalently</u>

B) $\vec{x}_1$ and $\vec{x}_2$ are <u>linearly independent</u>

Read 5.1 $\qquad \mathbb{A}+\mathbb{B}=? \qquad c\mathbb{A}+b\mathbb{B}=?$ easy

$$\underset{\uparrow \atop n\times m}{\mathbb{A}}\cdot\underset{\uparrow \atop m\times k}{\mathbb{B}} = \underbrace{\begin{bmatrix} \text{row } 1 \\ \text{row } 2 \\ \vdots \\ \text{row } n \end{bmatrix}}_{m\text{-vectors}} \begin{bmatrix} \underset{\uparrow \atop m\text{-vectors too}}{\text{col } 1}, & \text{col } 2, & \ldots, & \text{col } k \end{bmatrix}$$

$=$



$$c_{ij} = (\text{row } i)\cdot(\text{col } j)$$

Inverse of a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\underset{\mathbb{A}}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \cdot \underbrace{\begin{bmatrix} \vec{x}_1 & \vec{x}_2 \end{bmatrix}}_{\mathbb{A}^{-1}} = \underset{\mathbb{I}}{\mathbb{I}}$$