Lecture 25 Linear homogeneous systems with constant coefficients 5.2

Mon: MyLab HW 24

Wed: MyLab HW 25, GS HW 23, 24, 25

$$\underset{n\times m}{\mathbb{A}}\;\underset{m\times k}{\mathbb{B}} = \underset{n\times k}{\mathbb{C}}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 12 & 18 \\ 15 & 30 & 45 \end{bmatrix}$$

row 2 | col 3 | $c_{2,3}$

Cramer's rule:

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\mathbb{A}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{cases} x_1 = \dfrac{\det\begin{pmatrix} A & b \\ B & d \end{pmatrix}}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \dfrac{dA - bB}{\det} \\[2em] x_2 = \dfrac{\det\begin{pmatrix} a & A \\ c & B \end{pmatrix}}{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \dfrac{-cA + aB}{\det} \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\frac{1}{\det}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}_{\mathbb{A}^{-1}} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\mathbb{A}\vec{x} = \vec{b}$$

$$\vec{x} = \mathbb{A}^{-1}\vec{b}$$

Facts: $\mathbb{A}^{-1}\mathbb{A} = \mathbb{I}$ ✓

Interesting fact: $\mathbb{A}\mathbb{A}^{-1} = \mathbb{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ too!

surprising because $\underbrace{\mathbb{A}\mathbb{B}}_{\text{square matrices.}} \neq \mathbb{B}\mathbb{A}$ more often than $=$.

$\mathbb{A}$ square: $\mathbb{A}^{-1}$ exists $\Longleftrightarrow \det(\mathbb{A}) \neq 0$.

Complex e. vals

EX
$$\begin{cases} \dfrac{dx_1}{dt} = -2x_1 + 2x_2 \\[2ex] \dfrac{dx_2}{dt} = -2x_1 - 2x_2 \end{cases}$$

Linear homog
2x2 system
with const.
coeff.

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \vec{x}' = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} \qquad \mathbb{A} = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}$$

$$\boxed{\vec{x}' = \mathbb{A}\vec{x}}$$

Try sol^n $\vec{x} = \vec{a}e^{rt}$. Need $\mathbb{A}\vec{a} = r\vec{a}$

$$\boxed{(\mathbb{A} - r\mathbb{I})\vec{a} = \vec{0}} \quad \#2$$

For $\vec{a} \neq \vec{0}$: $\boxed{\det(\mathbb{A} - r\mathbb{I}) = 0}$ #1

$$\det \begin{bmatrix} -2-r & 2 \\ -2 & -2-r \end{bmatrix} = (r+2)^2 + 4 = 0$$

$$(r+2)^2 = -4$$

$$r+2 = \pm 2i$$

$$r = -2 \pm 2i$$

Euler: No problem!

For $r = -2+2i$: $(\mathbb{A} - r\mathbb{I})\vec{a} = \vec{0}$

$$\left[\begin{array}{cc|c} -2-(-2+2i) & 2 & 0 \\ -2 & -2-(-2+2i) & 0 \end{array}\right]$$

$$(-i)\cdot(\text{Row } 1) \longrightarrow \left[\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Trick: $\left[\begin{array}{cc|c} A & B & 0 \\ \sim\sim\sim & & 0 \end{array}\right]$

sol^n $\vec{a} = \begin{pmatrix} B \\ -A \end{pmatrix}$ or $\begin{pmatrix} -B \\ A \end{pmatrix}$

$\begin{pmatrix} i \\ -1 \end{pmatrix}$ or $\begin{pmatrix} -i \\ 1 \end{pmatrix}$

Old fashioned way: $\left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right]$

a_1 bound var.

a_2 free var.

$$a_1 + i\,a_2 = 0$$

Let $a_2 = c$, arbitrary. Get $a_1 = -i a_2 = -ic$

$$\text{Sol}^n \quad \vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -ic \\ c \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix} c$$

$\vec{a} = \begin{pmatrix} -i \\ 1 \end{pmatrix} c$ for $c \neq 0$ are all possible e.vects. for $r = -2+2i$.

Hmmm. $c = i$ gives me a nice one: $\begin{pmatrix} 1 \\ i \end{pmatrix}$

$$\text{Get} \quad \vec{x} = \vec{a}\, e^{rt} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-2+2i)t}$$

is a complex sol^n.

Fact: One complex sol^n gives two real sol^{n}s.

How $\begin{pmatrix} 1 \\ i \end{pmatrix} e^{-2t} \cdot e^{2it}$

Step 1

$$\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]\left(e^{-2t}\cos 2t + i e^{-2t}\sin 2t\right)$$

Step 2

$$\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t}\cos 2t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}\sin 2t\right] + i\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t}\sin 2t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}\cos 2t\right]$$

$i^2 = -1$

real & imag parts are real sol^{n}s

Why $\vec{x} = \vec{u} + i\vec{v}$ is a complex sol^n means

$$\vec{x}\,' = (\vec{u} + i\vec{v})' = \underline{\underline{\vec{u}\,'}} + i\underline{\underline{\vec{v}\,'}}$$

$$= A\vec{x} = A(\vec{u} + i\vec{v}) = \underline{\underline{A\vec{u}}} + i\underline{\underline{A\vec{v}}}$$

Are $\vec{x}_1 = \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} e^{-2t}$ $\quad \vec{x}_2 = \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^{-2t}$

a basis for gen^l sol^n ?

Same ques: $\begin{cases} \text{Are they linearly indep?} \\ \text{Is the Wronskian non-zero?} \end{cases}$

$$W[\vec{x}_1, \vec{x}_2] = \det\left[\vec{x}_1 , \vec{x}_2 \right]$$

$$= \det \begin{bmatrix} e^{-2t}\cos 2t & e^{-2t}\sin 2t \\ -e^{-2t}\sin 2t & e^{-2t}\cos 2t \end{bmatrix}$$

$$= e^{-2t}\cdot e^{-2t} \det \begin{bmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{bmatrix}$$

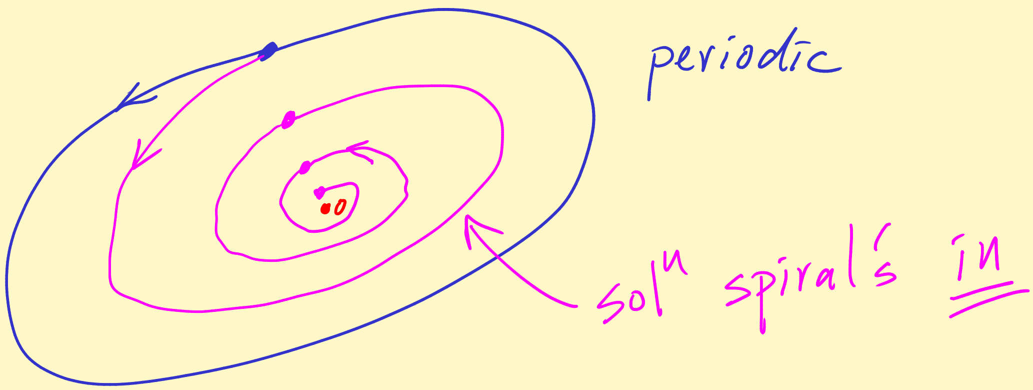
$$= e^{-4t}\left(\underbrace{\cos^2 2t + \sin^2 2t}_{\equiv 1} \right) = \underset{\uparrow \text{ never zero!}}{e^{-4t}}$$

Lazy way: Just check at $t=0$ because of Abel's.

$$W(0) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \neq 0 \quad ✓$$

Gen^l sol^n is $\quad \vec{x} = c_1 \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^{-2t}$

$$= \underbrace{\begin{pmatrix} \text{Vector of} \\ \text{sines \& cosines} \end{pmatrix}}_{\text{periodic}} \underset{\nwarrow \text{ dies away}}{e^{-2t}}$$



$\vec{x} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an "equillibrium" sol^n.

It is an asymptotically stable equillib.

Fun question: Is the spiral clockwise or counterclockwise?