Lecture 26 Complex eigenvalues 5.2 (part 2)

MyLab HW 25
GS HW 23, 24, 25

Last time:
$$\begin{cases} \dfrac{dx_1}{dt} = -2x_1 + 2x_2 \\ \dfrac{dx_2}{dt} = -2x_1 - 2x_2 \end{cases} \qquad \vec{x}\,' = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} \vec{x}$$

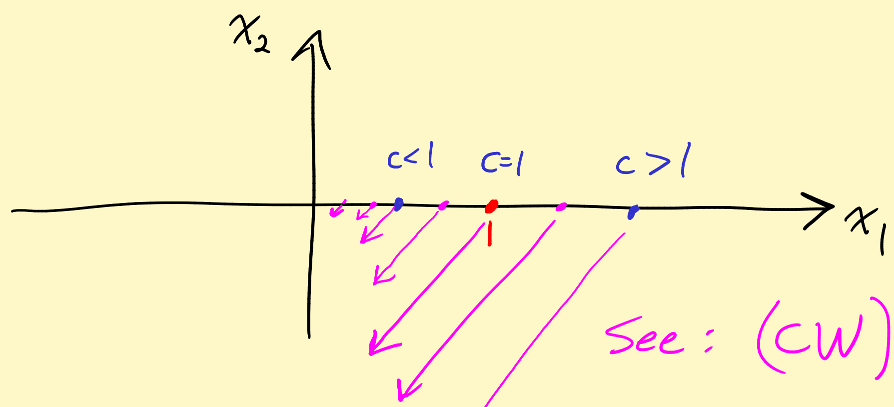
E. vals: $r = -2 \pm 2i$ Complex e.vect for $r = -2+2i$: $\vec{q} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

Complex sol^n: $\begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-2+2i)t}$

$$\vec{x} = c_1 \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^{-2t}$$

Spirals in. Clocwise (CW) or Counterclockwise? (CCW)

Plan: Test direction field: $\vec{x}\,' = A\vec{x}$ along $\mathbb{R}^+$.



$$\vec{x}\,' = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} \begin{pmatrix} c \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \end{pmatrix} c = \begin{pmatrix} -2c \\ -2c \end{pmatrix}$$

downward since negative

Spiral rule

negative: CW
positive: CCW

In vs. Out?

EX: Complex sol^n $\begin{pmatrix} 3+4i \\ 5+6i \end{pmatrix} e^{(2+3i)t}$

e.val $2+3i$

e.vect $\begin{pmatrix} 3+4i \\ 5+6i \end{pmatrix}$

Getting real sol^{n}s

$$\left[\begin{pmatrix}3\\5\end{pmatrix}+\begin{pmatrix}4\\6\end{pmatrix}i\right]\left(e^{2t}\cos 3t + i e^{2t}\sin 3t\right)$$

$$= \left[\underbrace{\begin{pmatrix}3\\5\end{pmatrix}e^{2t}\cos 3t \underset{i^2=-1}{-} \begin{pmatrix}4\\6\end{pmatrix}e^{2t}\sin 3t}_{\vec{x}_1}\right] + i\left[\underbrace{\begin{pmatrix}3\\5\end{pmatrix}e^{2t}\sin 3t + \begin{pmatrix}4\\6\end{pmatrix}e^{2t}\cos 3t}_{\vec{x}_2}\right]$$

Spirals out because of e^{2t}.

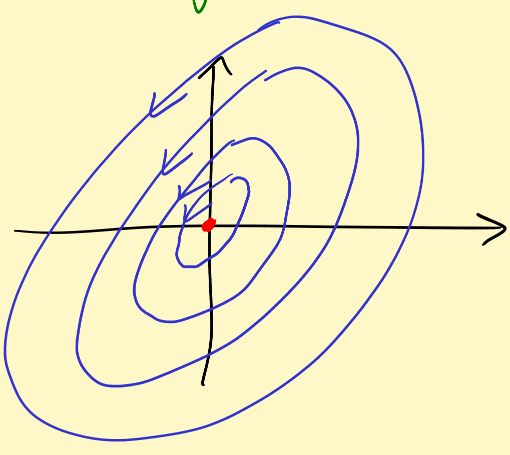
Fact Complex e.vals $r = a \pm bi$

$a > 0$ Spirals out (Unstable "source")

$a < 0$ Spirals in (Asymptotically stable "sink")

$a = 0$ "Center" (stable, but not asymp. stable)

Fact: $r = \pm bi$ is a "center". Trajectories in the "phase plane" are ellipses.



3X3 system

$$\begin{cases} \dfrac{dx_1}{dt} = 4x_2 \\[2mm] \dfrac{dx_2}{dt} = -x_1 \\[2mm] \dfrac{dx_3}{dt} = x_1 + 4x_2 - x_3 \end{cases}$$

3X3 linear homog system with const coeff

$$\vec{x}' = \begin{bmatrix} 0 & 4 & 0 \\ -1 & 0 & 0 \\ 1 & 4 & -1 \end{bmatrix} \vec{x} \qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Try $\vec{x} = \vec{a} e^{rt}$. Need $\det(\mathbb{A} - r\mathbb{I}) = 0$ #1

$(\mathbb{A} - r\mathbb{I})\vec{a} = \vec{0}$ #2

#1

$$\det \begin{bmatrix} 0-r & 4 & 0 \\ -1 & 0-r & 0 \\ 1 & 4 & -1-r \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= (+1) \cdot 0 \cdot \det \begin{bmatrix} -1 & -r \\ 1 & 4 \end{bmatrix} + (-1) \cdot 0 \cdot \det \begin{bmatrix} -r & 4 \\ 1 & 4 \end{bmatrix} + (+1) \cdot (-1-r) \det \begin{bmatrix} -r & 4 \\ -1 & -r \end{bmatrix}$$

$$= (-1-r)\left[r^2 + 4\right] = -(r+1)(r^2+4) = 0$$

e. vals $r = -1, \pm 2i$

<u>For $r = -1$</u>

$$\left[\begin{array}{ccc|c} 0-(-1) & 4 & 0 & 0 \\ -1 & 0-(-1) & 0 & 0 \\ 1 & 4 & -1-(-1) & 0 \end{array}\right]$$

$$(\mathbb{A} - r\mathbb{I})\vec{a} = \vec{0}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \end{array}\right]$$

Bound

$$\left[\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R1 \\ R2 + R1 \\ R3 - R1 \end{array}$$

a_1 bound $\quad$ a_2 bound $\quad$ a_3 free var

Let $a_3 = c$.

$$a_1 + 4a_2 = 0 \qquad a_1 = 0 \text{ too}$$

$$5a_2 = 0. \leftarrow a_2 = 0$$

$$\vec{a} = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} c \quad , \quad c \neq 0.$$

Take $c = 1$. Get $\vec{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-t}$

For $r = 2i$

$$\left[\begin{array}{ccc|c} 0-2i & 4 & 0 & 0 \\ -1 & 0-2i & 0 & 0 \\ 1 & 4 & -1-2i & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2i & 0 & 0 \\ -1 & -2i & 0 & 0 \\ 1 & 4 & -1-2i & 0 \end{array}\right] \quad R1/(-2i)$$

$$\left[\begin{array}{ccc|c} 1 & 2i & 0 & 0 \\ 0 & 4-2i & -1-2i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R3-R1 \\ R2-R1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2i & 0 & 0 \\ 0 & 1 & -i/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R2/(4-2i)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -i/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R1-2i\,R2$$

free

$$a_1 = a_3$$
$$a_2 = \frac{i}{2} a_3$$

Let $a_3 = c$.

$$\begin{cases} a_1 = a_3 = c \\ a_2 = \frac{i}{2} a_3 = \frac{i}{2} c \\ a_3 = c \end{cases}$$

$$\vec{a} = \begin{pmatrix} 1 \\ i/2 \\ 1 \end{pmatrix} c$$

Hmmm. Take $c=2$ to clear fraction.

Get $\vec{a} = \begin{pmatrix} 2 \\ i \\ 2 \end{pmatrix}$