Lecture 27 5.5 Repeated roots

MyLab HW 26

Exam 2 Tues, April 7 Elliott 6:30pm

Last time
$$\begin{cases} \frac{dx_1}{dt} = 4x_2 \\ \frac{dx_2}{dt} = -x_1 \\ \frac{dx_3}{dt} = x_1 + 4x_2 - x_3 \end{cases}$$

$$\vec{x}\,' = \begin{bmatrix} 0 & 4 & 0 \\ -1 & 0 & 0 \\ 1 & 4 & -1 \end{bmatrix} \vec{x}$$

e.vals $r = -1, \pm 2i$

For $r=-1$ Got $\vec{a} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. $\vec{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-t}$

For $r = 2i$
$$\left[\begin{array}{ccc|c} 0-2i & 4 & 0 & 0 \\ -1 & 0-2i & 0 & 0 \\ 1 & 4 & -1-2i & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -i/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

corrected from last time

free

$a_1 = a_3$

$a_2 = \frac{i}{2} a_3$

Let $a_3 = c$.

Next thing: Make a list

$$\begin{cases} a_1 = a_3 = c \\ a_2 = \frac{i}{2} a_3 = \frac{i}{2} c \\ a_3 = c \end{cases}$$

$$\vec{a} = \begin{pmatrix} c \\ \frac{i}{2}c \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ i/2 \\ 1 \end{pmatrix} c, \quad c \neq 0$$

can take $c \in \mathbb{C}$, $c \neq 0$

Pick a nice one! Take $c = 2$.

$$\vec{u} = \begin{pmatrix} 2 \\ i \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} i$$

Complex sol^n: $\begin{pmatrix} 2 \\ i \\ 2 \end{pmatrix} e^{2it}$

$$= \left[\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} i \right] (\cos 2t + i \sin 2t)$$

$$= \Bigg[\underbrace{\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \cos 2t - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \sin 2t}_{\vec{x}_2} \Bigg] + i \Bigg[\underbrace{\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \sin 2t + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cos 2t}_{\vec{x}_3} \Bigg]$$

$i^2 = -1$

Gen^l Sol^n

$$\vec{x} = c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2\cos 2t \\ -\sin 2t \\ 2\cos 2t \end{pmatrix} + c_3 \begin{pmatrix} 2\sin 2t \\ \cos 2t \\ 2\sin 2t \end{pmatrix}$$

$$W(0) = \det \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} = (+1) \cdot (1) \det \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= 2 \neq 0 \checkmark$$

Repeated roots

(Dumb) example:
$$\begin{cases} \dfrac{dx_1}{dt} = 2x_1 \\[2ex] \dfrac{dx_2}{dt} = 2x_2 \end{cases}$$

independent!

$$\vec{x}' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \vec{x}$$

$$\det(\mathbb{A} - r\,\mathbb{I}) = 0$$

$$\det \begin{bmatrix} 2-r & 0 \\ 0 & 2-r \end{bmatrix} = 0$$

$$(2-r)^2 = 0$$

$r = 2, 2$ (repeated)

<u>For $r=2$</u> $\quad (\mathbb{A} - \underbrace{r}_{r=2}\,\mathbb{I})\vec{a} = \vec{0}$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad !$$

What? a_1 and a_2 can both be anything!

They are both <u>free</u> variables.

Let $a_1 = s$

$a_2 = t$

$$\vec{a} = \begin{pmatrix} s \\ t \end{pmatrix} = s\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

<u>two</u> linearly ind. e-vects

Get <u>two</u> linearly ind sol^{n}s from $\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}$.

[$W(0) = 1 \neq 0$ ✓] $\quad$ Gen^{l} sol^{n}: $c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2t}$

Not dumb example: $\vec{x}\,' = \begin{pmatrix} 1 & -3 \\ 3 & 7 \end{pmatrix} \vec{x}$

$$\det \begin{pmatrix} 1-r & -3 \\ 3 & 7-r \end{pmatrix} = (1-r)(7-r) + 9$$
$$= r^2 - 8r + 16$$
$$= (r-4)^2 = 0$$

$r = 4, 4.$

<u>For $r = 4$</u>

$$\left[\begin{array}{cc|c} 1-4 & -3 & 0 \\ 3 & 7-4 & 0 \end{array}\right] \quad (A - 4I)\vec{a} = \vec{0}$$

$$\left[\begin{array}{cc|c} -3 & -3 & 0 \\ 3 & 3 & 0 \end{array}\right]$$

a_1 bound

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

a_2 free

$a_1 = -a_2 = -c$

$a_2 = c$

$$\vec{a} = \begin{pmatrix} -c \\ c \end{pmatrix}, \quad c \neq 0$$

Pick $c=-1$. $\vec{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ Ouch! Only get one sol^n: $\vec{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}$

Need $\vec{x}_2$ ind. sol^n.

Hmmm. Try $\vec{x}_2 = \vec{a}\, t e^{rt}$. Doesn't work!

Method: Try $\boxed{\vec{x} = \vec{a}\, t e^{rt} + \vec{b} e^{rt}}$ ← $r = $ e.val

Need $\vec{x}\,' = A\vec{x}$

$$\left[\left(\vec{a} e^{rt} + r\vec{a}\, t e^{rt}\right) + r\vec{b} e^{rt}\right] = A\left(\vec{a}\, t e^{rt} + \vec{b} e^{rt}\right)$$

Can divide out all e^{rt}'s.

$$\vec{a} + rt\,\vec{a} + r\vec{b} = A\vec{a}\,t + A\vec{b}$$

$$\underbrace{\vec{a} + r\vec{b} - A\vec{b}}_{\text{no } t\text{'s}} = \underbrace{A\vec{a}\,t - rt\,\vec{a}}_{t\text{'s}}$$

Hmmm. Set $t=0$. See LHS is $\vec{0}$.

Set $t=1$. See $A\vec{a} - r\vec{a} = \vec{0}$

Get 2 systems: $\boxed{A\vec{a} = r\vec{a}}$ $\boxed{(A - rI)\vec{b} = \vec{a}}$

$\vec{a} = $ e.vect for r Now find $\vec{b}$.

Fact $\vec{x}_2 = \vec{a}te^{rt} + \vec{b}e^{rt}$

where r = repeated e-val. $\vec{a}$ = e.vect for r.

and $\vec{b}$ satisfies $\boxed{(A - rI)\vec{b} = \vec{a}}$

Have $r = 4$, $\vec{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$$(A - 4I)\vec{b} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \leftarrow \vec{a}$$

$$\left[\begin{array}{cc|c} 1-4 & -3 & 1 \\ 3 & 7-4 & -1 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -3 & -3 & 1 \\ 3 & 3 & -1 \end{array}\right]$$

b_1 bound

$$\sim \left[\begin{array}{cc|c} 1 & 1 & -1/3 \\ 0 & 0 & 0 \end{array}\right]$$

b_2 free

Let $b_2 = c$

$$b_1 + b_2 = -\frac{1}{3}$$

$$b_1 = -b_2 - 1/3$$

List $\Big($ $b_2 = c$

$$\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} -c - 1/3 \\ c \end{pmatrix}$$ $\leftarrow$ pick nice one.

Take $c = 0$. $\quad \vec{b} = \begin{pmatrix} -1/3 \\ 0 \end{pmatrix}$

Get $\quad \vec{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{4t} + \begin{pmatrix} -1/3 \\ 0 \end{pmatrix} e^{4t}$

Gen^{l} Sol^{n} $\quad \underline{\vec{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} t - 1/3 \\ -1 \end{pmatrix} e^{4t}}$