Lecture 28 Pictures! 5.3

Mon: MyLab HW 27
Wed: MyLab HW 28, GS HW 26,27,28

$$\begin{cases} \dfrac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 \\ \dfrac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 \end{cases}$$

Abel's: $W = Ce^{\int (a_{11}+a_{22})\,dt}$

Try $\vec{x} = \vec{a}e^{rt}$ Need $\det(A - rI) = 0$. Get roots $r_1, r_2$.

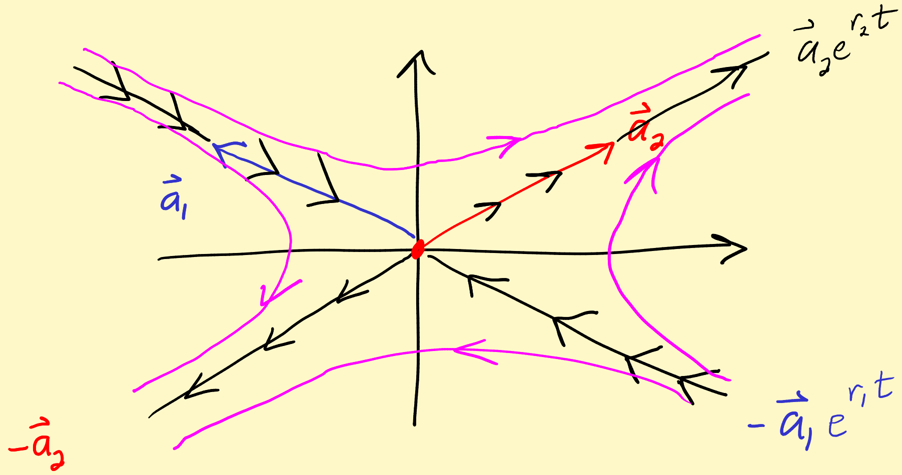
For each root: Get e-vect $\vec{a}_i$ with $(A - r_i I)\vec{a}_i = \vec{0}$

Cases 1) Roots $r_1, r_2$ are real and opposite sign:

$r_1 < 0 < r_2$. Get Gen$^l$ Sol$^n$:

$$\vec{x} = c_1\vec{a}_1 e^{r_1 t} + c_2 \vec{a}_2 e^{r_2 t}$$

Fact e-vects for distinct e-vals are always lin. indep.




$\begin{cases} c_2 = 1 \\ c_1 = 0 \end{cases}$ $\vec{a}_2 e^{r_2 t}$

$\begin{cases} c_2 = -1 \\ c_1 = 0 \end{cases}$ $-\vec{a}_2 e^{r_2 t}$

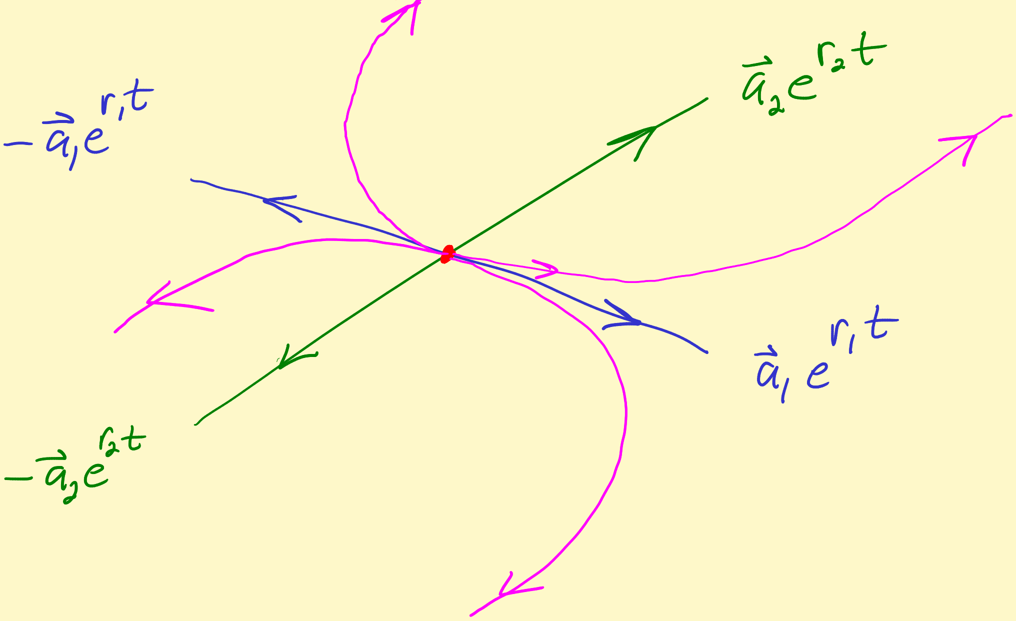
Saddle point. Unstabe.

Saddle rule: Trajectories in the plane come in asymptotically to $\pm$ e-vector direction corresponding to the negative e-val, and go out asympt. to $\pm$e-vector for the positive e-val.

Case $r_1, r_2$ real and unequal and same sign.

Subcase $0 < r_1 < r_2$

$r_1$ closest to zero




Why: $t \to -\infty$ $\quad \vec{x} = c_1 \vec{a}_1 e^{r_1 t} + c_2 \vec{a}_2 e^{r_2 t}$

much smaller than $e^{r_1 t}$

$$= e^{r_1 t} \left[ c_1 \vec{a}_1 + c_2 \underbrace{\frac{e^{r_2 t}}{e^{r_1 t}}}_{\to 0 \text{ as } t \to -\infty} \vec{a}_2 \right]$$

See that solutions appear to originate from the origin along $\pm \vec{a}_1$ directions.

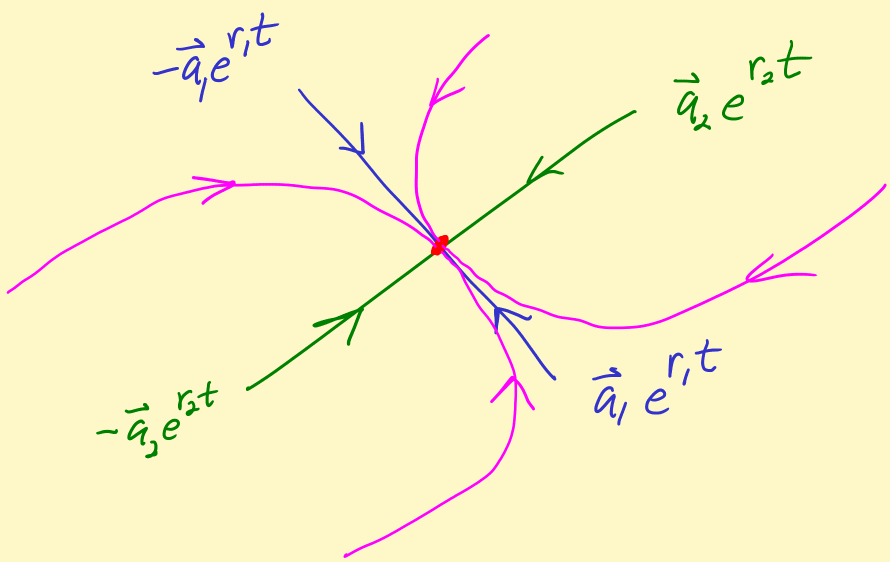
Improper node : Unstable. Source

Node rule: Near the origin, trajectories hug the $\pm$ e-vect direction corresponding to the e-val closest to zero.

Part 2 Far away, they look parallel to the other $\pm$ e-vect direction.

Subcase: $r_2 < r_1 < 0$

closest to zero



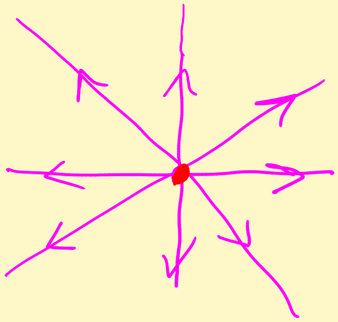

Node rule still holds! Same picture with arrows reversed.

Improper node. Asymptotically stable. Sink

EX: A proper node.

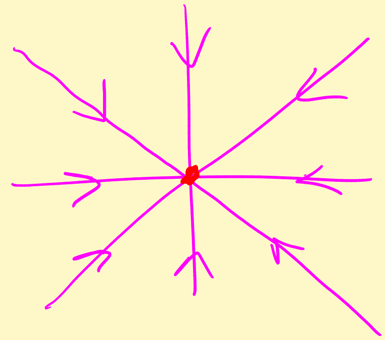
$$\begin{cases} \dfrac{dx_1}{dt} = k x_1 \\[2ex] \dfrac{dx_2}{dt} = k x_2 \end{cases}$$

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{kt} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{kt} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{kt}$$



$k > 0$

Unstable source.



$k < 0$

Asympt. stable sink.

<u>Case</u> $r = a \pm bi$. For $r = a + bi$, get complex e-vect. $\vec{A} + i\vec{B}$

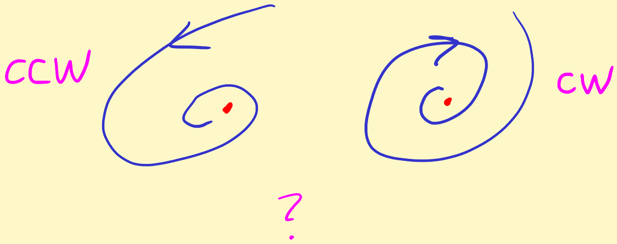
Complex sol$^n$ $\vec{x} = [\vec{A} + i\vec{B}](e^{at}\cos bt + i e^{at}\sin bt)$

$$= \Big(\underbrace{\vec{A}e^{at}\cos bt \underset{\substack{\uparrow \\ i^2=-1}}{-} \vec{B}e^{at}\sin bt}_{\vec{x}_1}\Big) + i\Big(\underbrace{\vec{A}e^{at}\sin bt + \vec{B}e^{at}\cos bt}_{\vec{x}_2}\Big)$$

Gen$^l$ sol$^n$: $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$

<u>Critical thing</u> Sign of $a = \text{Re}\, r$.

<u>Subcase</u> $a < 0$. Type: <u>Spiral</u> in

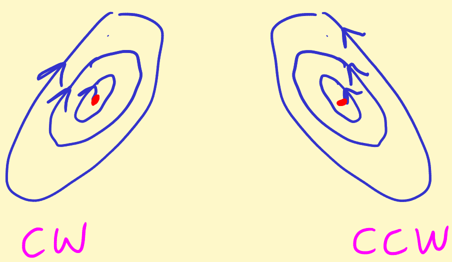



(in) <u>sink</u>

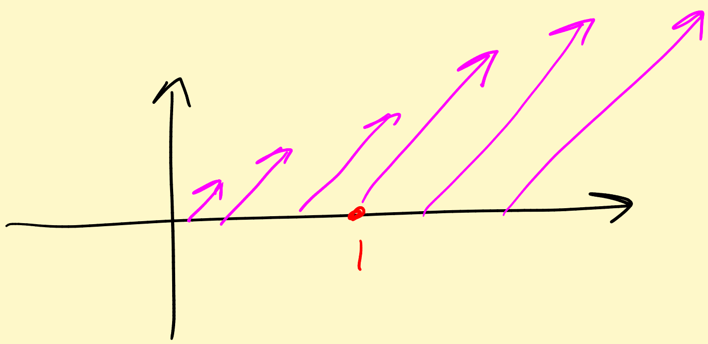
<u>Asympt. stable</u>

<u>Subcase</u> $a>0$ : Spiral <u>out</u>, <u>source</u>, <u>Unstable</u>.

<u>Subcase</u> $a=0$ : <u>Center</u>, <u>stable</u>, but <u>not</u> asymptotically stable.




CW or CCW test: Test field at $\begin{pmatrix}1\\0\end{pmatrix} \sim (1,0)$




$$\vec{x}'\Big|_{\binom{1}{0}} = A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a_{11}\\a_{21}\end{pmatrix}$$ ← sign of $a_{21}$ determines up or down.

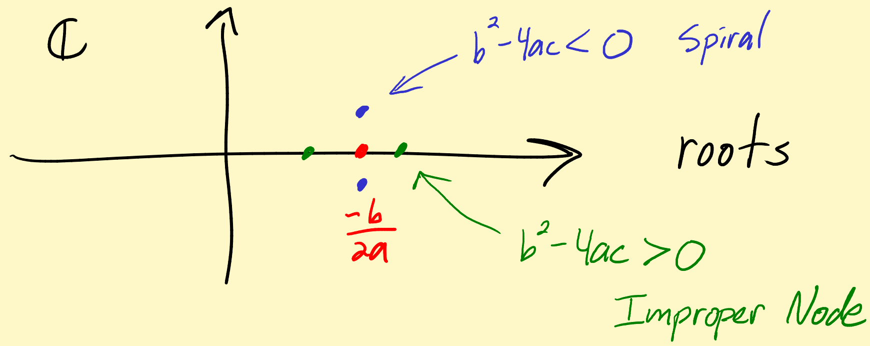
Spiral rule: $a_{21}>0$ : <u>up</u> CCW

$a_{21}<0$ : <u>down</u> CW

Case What about $r_1 = r_2$ case? Very delicate!

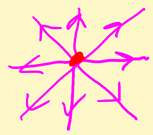
$$r = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$b^2 - 4ac = 0$



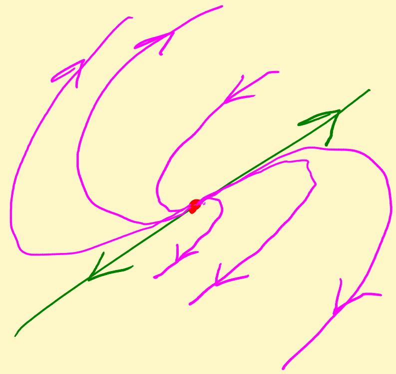

Subcase repeated roots, two lin indep e.vects (A non-defective)

Proper node.



Subcase Only one e-vect (A defective)

(Very) Improper Node



Fact $r = \frac{-b}{2a}$ repeated

$\frac{-b}{2a} > 0$ Unstable source

$\frac{-b}{2a} < 0$ A. Stable sink