Lecture 29 The exponential of a matrix & fundamental matrices 5.6

MyLab HW 28. Gradescope HW 26, 27, 28

MyLab HW 29 on Friday $\leftarrow$ Good to do because Lesson 29 on Exm 2

Fundamental matrices

$\vec{x}' = \mathbb{A}\vec{x}$, $\vec{x}(0) = \vec{x}_0$ $\mathbb{A}$ $n \times n$

Need n lin. ind. sol^{n}s $\vec{x}_j' = \mathbb{A}\vec{x}_j$, $j = 1, 2, \ldots, n$

Gen^l sol^n $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n$

$$= \underbrace{\left[\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\right]}_{\mathbb{X} = \mathbb{X}(t)} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

$\leftarrow$ Fundamental matrix

Note Wronskian $W = \det(\mathbb{X}) \neq 0$.

Hmmm.
$$\begin{cases} \vec{x}_1' = \mathbb{A}\vec{x}_1 \\ \vec{x}_2' = \mathbb{A}\vec{x}_2 \\ \vdots \\ \vec{x}_n' = \mathbb{A}\vec{x}_n \end{cases}$$

$$\underbrace{\left[\vec{x}_1', \ldots, \vec{x}_n'\right]}_{\mathbb{X}'\ !} = \left[\mathbb{A}\vec{x}_1, \ldots, \mathbb{A}\vec{x}_n\right]$$

$$= \mathbb{A}\underbrace{\left[\vec{x}_1, \ldots, \vec{x}_n\right]}_{\mathbb{X}}$$

Aha! $\mathbb{X}' = \mathbb{A}\mathbb{X}$

Looks familiar! Freshman calc: $y' = ay$

Sol^n $y = y(0)e^{at}$

Use $\mathbb{X}$ to solve IVP : $\vec{x}(0) = \vec{x}_0$.

$$\vec{x} = \mathbb{X}\vec{c}$$

Want $\vec{x}(0) = \underline{\mathbb{X}(0)\,\vec{c} = \vec{x}_0}$ $\leftarrow$ $W(0) = \det(\mathbb{X}(0)) \neq 0$

$\mathbb{X}(0)^{-1}$ exists

Multiply on left by $\mathbb{X}(0)^{-1}$:

Get $\boxed{\vec{c} = \mathbb{X}(0)^{-1}\vec{x}_0}$

Sol^n : $\vec{x} = \underbrace{\mathbb{X}\cdot\mathbb{X}(0)^{-1}}_{\text{Normalized Fund matrix } \Phi}\vec{x}_0$

EX A 2x2 Gen^l sol^n $\vec{x} = c_1\begin{pmatrix}5\\1\end{pmatrix}e^{3t} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{t}$

$$\mathbb{X} = \left[\begin{pmatrix}5\\1\end{pmatrix}e^{3t}, \begin{pmatrix}1\\-1\end{pmatrix}e^{t}\right] = \begin{bmatrix}5e^{3t} & e^{t}\\ e^{3t} & -e^{t}\end{bmatrix}$$

$$\mathbb{X}(0) = \begin{bmatrix}5 & 1\\ 1 & -1\end{bmatrix}$$

Nice formula to memorize : $\begin{bmatrix}a & b\\ c & d\end{bmatrix}^{-1} = \frac{1}{\det}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$

$$\mathbb{X}(0)^{-1} = \frac{1}{5\cdot(-1)-1}\begin{bmatrix}-1 & -1\\ -1 & 5\end{bmatrix} = \begin{bmatrix}1/6 & 1/6\\ 1/6 & -5/6\end{bmatrix}$$

Sol^n $\vec{x}$ with $\vec{x}(0) = \begin{pmatrix}6\\7\end{pmatrix}$ is $c_1\begin{pmatrix}5\\1\end{pmatrix}e^{3t} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{t}$

where $\quad \vec{c} = X(0)^{-1}\begin{pmatrix}6\\7\end{pmatrix} = \begin{bmatrix}1/6 & 1/6\\ 1/6 & -5/6\end{bmatrix}\begin{pmatrix}6\\7\end{pmatrix} = \begin{pmatrix}13/6\\ -29/6\end{pmatrix}$

$c_1 = \frac{13}{6}, \quad c_2 = \frac{-29}{6}$

Normalized Fund matrix For extra lazy people!

Makes solving IVP's a snap.

Want a fund matrix with easy $X(0)^{-1}$. How about I.

(2x 2)
$$\left[\vec{x}_1(0), \vec{x}_2(0)\right] = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$
$$\uparrow \qquad \uparrow$$
$$\vec{e}_1 \qquad \vec{e}_2$$

Want sol^{n}s with $\vec{x}_1(0) = \vec{e}_1, \quad \vec{x}_2(0) = \vec{e}_2$

How to find them. Say we have $X = [\vec{x}_1, \vec{x}_2]$.

Want new ones $\vec{\chi}_1, \vec{\chi}_2$ with
$$\vec{\chi}_1 = X\vec{c}_1$$
$$\vec{\chi}_2 = X\vec{c}_2$$

$$\underbrace{\left[\vec{\chi}_1(0), \vec{\chi}_2(0)\right]}_{I} = \left[X(0)\vec{c}_1, X(0)\vec{c}_2\right]$$

$$I = X(0)\cdot \underbrace{\left[\vec{c}_1, \vec{c}_2\right]}_{\text{must } = X(0)^{-1}}$$

Get $[\vec{c}_1, \vec{c}_2] = X(0)^{-1}$ and

$$\underbrace{[\vec{x}_1, \vec{x}_2]}_{\Phi} = X\,[\vec{c}_1, \vec{c}_2] = \underbrace{X\,X(0)^{-1}}_{\Phi}$$

Important formula $\boxed{\Phi = X\,X(0)^{-1}}$

Exciting thing: Sol^n to IVP $\vec{x}(0) = \vec{x}_0$ is

$$\Phi\,\vec{x}_0.$$

EX cont'd $X = \begin{bmatrix} 5e^{3t} & e^t \\ e^{3t} & -e^t \end{bmatrix}$ $X(0)^{-1} = \begin{bmatrix} 1/6 & 1/6 \\ 1/6 & -5/6 \end{bmatrix}$

$$\Phi = X\,X(0)^{-1} = \begin{bmatrix} \frac{5}{6}e^{3t} + \frac{1}{6}e^t & \frac{5}{6}e^{3t} - \frac{5}{6}e^t \\ \frac{1}{6}e^{3t} - \frac{1}{6}e^t & \frac{1}{6}e^{3t} + \frac{5}{6}e^t \end{bmatrix}$$

Check: $\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

Sol^n to IVP $\vec{x}(0) = \begin{pmatrix} 10 \\ 11 \end{pmatrix}$ is $\vec{x} = \Phi \cdot \begin{pmatrix} 10 \\ 11 \end{pmatrix}$

Mind blowing math:

$$y' = ay$$

$$y = y(0)\,e^{at}$$

$$e^{at} = 1 + \frac{at}{1!} + \frac{a^2t^2}{2!} + \dots$$

$$e^0 = 1$$

Hmmm:

$$\mathbb{X}' = \mathbb{A}\mathbb{X}$$

$$\boxed{\Phi' = \mathbb{A}\Phi \quad \text{with } \Phi(0) = \mathbb{I}}$$

Hmmm. $\Phi \overset{?}{=} \mathbb{I} + \frac{1}{1!}\mathbb{A}t + \frac{1}{2!}\mathbb{A}^2 t^2 + \frac{1}{3!}\mathbb{A}^3 t^3 + \cdots$

Miracle! It works! Uniqueness removes ?

<u>Def</u>

$\underline{\text{Def}^n}$ The matrix exponential $e^{\mathbb{A}t}$ is Φ.

$\underline{\text{Fun facts}}$ Possible for $\mathbb{A}^n$ = zero matrix (nilpotent).

$\underline{\text{EX}}$: $\mathbb{A} = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$ $\qquad \mathbb{A}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$!

$$\mathbb{A}^n = \mathbb{O} \quad n \geq 2.$$

Great $\quad e^{\mathbb{A}t} = \mathbb{I} + \mathbb{A}t + \mathbb{O} + \mathbb{O} + \cdots$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} t = \begin{bmatrix} 1-t & -t \\ t & 1-t \end{bmatrix}$$

Done!

$\underline{\text{Fact \#2}}$ Diagonal matrices are easy

$$\mathbb{D} = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}$$

$$e^{\mathbb{D}t} = \begin{bmatrix} e^{d_1 t} & & 0 \\ & e^{d_2 t} & \\ 0 & & \ddots & \\ & & & e^{d_n t} \end{bmatrix}$$

Fact #3 $\quad e^{(\mathbb{A}+\mathbb{B})t} \overset{?}{=} e^{\mathbb{A}t} e^{\mathbb{B}t}$

Yes, when $\mathbb{A}, \mathbb{B}$ commute:

$$\mathbb{A}\mathbb{B} = \mathbb{B}\mathbb{A}$$

Fact #4 Diagonal matrices commute with anything.

HWK prob $\quad \mathbb{A} = \underbrace{\mathbb{D}}_{\text{diag}} + \underbrace{\mathbb{N}}_{\text{nilpotent}}$

$$e^{\mathbb{A}t} = \underbrace{e^{\mathbb{D}t}}_{\text{easy}} \; \underbrace{e^{\mathbb{N}t}}_{\text{easy too}}$$