Lecture 30 Non-homogeneous linear systems, Undet. coeffs., Var. of Par. 5.7

MyLab HW 29

Exam 2 Tues, 6:30 pm on Lessons 17–29.

EX

$$(\dagger)\begin{cases} \dfrac{dx_1}{dt} = p_{11}(t)\,x_1 + p_{12}(t)\,x_2 + g_1(t) \\[2ex] \dfrac{dx_2}{dt} = p_{21}(t)\,x_1 + p_{22}(t)\,x_2 + g_2(t) \end{cases}$$

non-homog

$\vec{x}\,' = \mathbb{P}(t)\vec{x} + \vec{g}$

Wronskian: $W = Ce^{\int p_{11}(t) + p_{22}(t)\,dt}$ ← Abel's formula

Homogeneous system: $\vec{x}\,' = \mathbb{P}(t)\,\vec{x}$ $(*)$

Step 1 Solve homog sys $(*)$. Get $\vec{x}_c = c_1\vec{x}_1 + c_2\vec{x}_2$

Step 2 Get one particular sol^n $\vec{x}_p$ to $(\dagger)$.

Fact General sol^n to $(\dagger)$ is

$$\vec{x} = \underbrace{\left(c_1\vec{x}_1 + c_2\vec{x}_2\right)}_{\text{Gen' sol}^n\text{ to homog}} + \vec{x}_p$$

↑ particular sol^n

Why:

$$\vec{x}_p\,' = \mathbb{P}\,\vec{x}_p + \vec{g}$$

Suppose $\vec{x}$ also solves $(\dagger)$.

$$\vec{x}\,' = \mathbb{P}\,\vec{x} + \vec{g}$$

(Bottom − top)

$$(\vec{x} - \vec{x}_p)' = \mathbb{P}(\underbrace{\vec{x} - \vec{x}_p}_{\vec{u}}) + (\vec{g} - \vec{g})$$

$\vec{u}\,' = \mathbb{P}\vec{u}$. $\vec{u}$ solves $(*)$. So

$\vec{x} - \vec{x}_p = c_1 \vec{x}_1 + c_2 \vec{x}_2$ for some c_1 & c_2. ✓

One more thing to show: Things of form $c_1\vec{x}_1 + c_2\vec{x}_2 + \vec{x}_p$ all solve (+). Easy part. ✓

EX $\quad \vec{x}\,' = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \vec{x} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$

Step 1 $\quad$ Homog sol^n : $\quad \vec{x}_c = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}$

Step 2 $\quad$ 2 is not e.val, so the "Method of undet. coeff." is the tool. $\boxed{\text{Guess } \vec{x}_p = \vec{b}\, e^{2t}}$

Force it! $\quad$ Plug in $\quad \vec{x}_p\,' = 2\vec{b}\, e^{2t}$

$$\vec{x}_p\,' = \mathbb{A}\,\vec{x}_p + \vec{g}$$

$$2\vec{b}\,e^{2t} \overset{\text{want}}{=} \mathbb{A}\,\vec{b}\,e^{2t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$$

cancel out all e^{2t}'s

$$2\vec{b} = \mathbb{A}\,\vec{b} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

OK to do
$$\begin{cases} 2b_1 = -3b_1 + b_2 + 1 \\ 2b_2 = b_1 - 3b_2 + 2 \end{cases}$$
Solve for b_1, b_2

or $\quad \mathbb{A}\,\vec{b} - 2\vec{b} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

$$(\mathbb{A} - 2\mathbb{I})\vec{b} = \begin{pmatrix}-1\\-2\end{pmatrix}$$

$$\begin{bmatrix}-3-2 & 1\\ 1 & -3-2\end{bmatrix}\vec{b} = \begin{pmatrix}-1\\-2\end{pmatrix}$$

$$\left[\begin{array}{cc|c}-5 & 1 & -1\\ 1 & -5 & -2\end{array}\right]$$

Row operations, or Cramer's rule for 2x2

$$b_1 = \frac{\det\begin{bmatrix}-1 & 1\\-2 & -5\end{bmatrix}}{\det\begin{bmatrix}-5 & 1\\1 & -5\end{bmatrix}} = \frac{7}{24}$$

$$b_2 = \frac{11}{24}$$

So $\quad \vec{x}_p = \vec{b}e^{2t} = \frac{1}{24}\begin{pmatrix}7\\11\end{pmatrix}e^{2t}$

Step 3 $\quad$ Gen' sol^n $= c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-2t} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{-4t} + \frac{1}{24}\begin{pmatrix}7\\11\end{pmatrix}e^{2t}$

Method of Undet. Coeff

$$\vec{x}\,' = \underset{\substack{\uparrow\\\text{const coeff}}}{\mathbb{A}}\vec{x} + \underset{\substack{\uparrow\\\text{special form}}}{\vec{g}}$$

$\vec{g}$	Try $\vec{x}_p =$
$\begin{pmatrix}3\\4\end{pmatrix}e^{rt}$	$\vec{b}e^{rt}$ $\leftarrow$ danger if r is e.val for $\mathbb{A}$

$\binom{2}{3}t^2$	$t^2\vec{b}_2 + t\vec{b}_1 + \vec{b}_0$ ← danger when e.val $r=0$
$\binom{7}{8}\cos\omega t$ or $\binom{9}{10}\sin\omega t$	$\vec{a}\cos\omega t + \vec{b}\sin\omega t$ ← danger e.val $\pm\omega i$
etc.	

When danger, use Method of Variation of Parameters

$$\vec{x}\,' = \mathbb{P}\vec{x} + \vec{g}$$ ← $\mathbb{P}$ can be $\mathbb{P}(t)$, $\vec{g}$ can be anything

Step 1 Get homog sol^{n} $\vec{x}_c = c_1\vec{x}_1 + c_2\vec{x}_2$

Step 2 Look for $\vec{x}_p = u_1\vec{x}_1 + u_2\vec{x}_2$

fcns

$$\vec{x}_p\,' = u_1'\vec{x}_1 + u_1\vec{x}_1\,' + u_2'\vec{x}_2 + u_2\vec{x}_2\,'$$

$$\overset{\text{want}}{=} \mathbb{P}(u_1\vec{x}_1 + u_2\vec{x}_2) + \vec{g}$$

$$= u_1\mathbb{P}\vec{x}_1 + u_2\mathbb{P}\vec{x}_2 + \vec{g}$$

cancel!

$$\boxed{u_1'\vec{x}_1 + \vec{u}_2\,'\vec{x}_2 = \vec{g}}$$

$$\left[\vec{x}_1 \, , \, \vec{x}_2\right]\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \vec{g}$$

$$\boxed{\mathbb{X}\,\vec{u}\,' = \vec{g}} \leftarrow \text{formula to memorize}$$

or

$$\vec{u}\,' = \mathbb{X}^{-1}\,\vec{g}$$

EX $\quad \vec{x}\,' = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}\vec{x} + \begin{pmatrix} 1 \\ e^t \end{pmatrix} \leftarrow e^t\,!$

Step 1 $\quad$ Homog sol^n $\quad c_1\begin{pmatrix}1\\1\end{pmatrix}e^t + c_2\begin{pmatrix}3\\2\end{pmatrix}e^{2t} \leftarrow$ danger! $r=1$ is e.val

Step 2 $\quad$ Var of Par $\quad \mathbb{X} = \left[\begin{pmatrix}1\\1\end{pmatrix}e^t,\ \begin{pmatrix}3\\2\end{pmatrix}e^{2t}\right]$

$$\mathbb{X} = \begin{bmatrix} e^t & 3e^{2t} \\ e^t & 2e^{2t} \end{bmatrix} \qquad \det(\mathbb{X}) = -e^{3t}$$

$$\mathbb{X}^{-1} = \frac{1}{(-e^{3t})}\begin{bmatrix} 2e^{2t} & -3e^{2t} \\ -e^t & e^t \end{bmatrix}$$

$$\vec{u}\,' = \mathbb{X}^{-1}\,\vec{g} = \mathbb{X}^{-1}\begin{pmatrix}1\\e^t\end{pmatrix}$$

$$\begin{pmatrix}u_1'\\u_2'\end{pmatrix} = \vec{u}\,' = \frac{-1}{e^{3t}}\begin{pmatrix} 2e^{2t}-3e^{3t} \\ -e^t+e^{2t} \end{pmatrix} = \begin{pmatrix} -2e^{-t}+3 \\ e^{-2t}-e^{-t} \end{pmatrix}$$

!

Need to integrate!

$$u_1 = \int -2e^{-t} + 3 \, dt = 2e^{-t} + 3t$$

$$u_2 = \int e^{-2t} - e^{-t} \, dt = -\tfrac{1}{2}e^{-2t} + e^{-t}$$

no +C's

Finally $\vec{x}_p = u_1 \vec{x}_1 + u_2 \vec{x}_2$

<u>Fun fact</u> $y'' + p(t)\,y' + q(t)\,y = R(t)$

Turn into 2x2 system: $\begin{cases} x_1 = y \\ x_2 = y' \end{cases}$

$x_1' = y' = x_2 \leftarrow$ Eqn 1

$x_2' = y'' = -q(t)\,y - p(t)\,y' + R(t)$

$= -q\,x_1 - p\,x_2 + R \leftarrow$ Eqn 2

$$\boxed{\vec{x}\,' = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \vec{x} + \begin{pmatrix} 0 \\ R \end{pmatrix}}$$

Var of Par: $\begin{cases} u_1' y_1 + u_2' y_2 = 0 \leftarrow \text{Didn't have to "make it up"} \\ u_1' y_1' + u_2' y_2' = R \end{cases}$