Review for Exam 2 Tues, April 7, 6:30–7:30 pm, Elliott

Covers Lessons 17–29. Only grade front side of paper. Back for scratch.
1–7 multiple choice; 8, 9, 10 written answers. Practice problems at
www.math.purdue.edu/MA266 (Problems 17–27)

17. A mass-spring-dashpot system with mass $m = 4$, damping constant $c = 4$, and spring constant $k = 17$ is set in free motion with initial conditions $x(0) = 0$ and $x'(0) = 2$, where $x(t)$ is the displacement from the equilibrium position at time t. Find $x(2)$.

A. $e^{-1}\sin(4)$ B. $e^{-1}\cos(4)$ C. $e^{-1}\cos(4) + e^{-1}\sin(4)$ D. $e^{-2}\cos(1) - e^{-2}\sin(1)$ E. $e^{-2}\sin(1)$

$$m\underset{\uparrow\,\ddot{x}}{a} = -kx - c\dot{x}$$

$$4\ddot{x} + 4\dot{x} + 17x = 0$$

$$4r^2 + 4r + 17 = 0$$

$$r = \frac{-4 \pm \sqrt{16 - 16\cdot 17}}{8} = -\frac{1}{2} \pm \frac{4\sqrt{-16}}{8} = -\frac{1}{2} \pm 2i$$

$$x = c_1 e^{-\frac{1}{2}t}\cos 2t + c_2 e^{-\frac{1}{2}t}\sin 2t \qquad x(0) = c_1 \overset{\text{want}}{=} 0$$

$$\dot{x} = -\frac{1}{2}c_1 e^{-\frac{1}{2}t}\cos 2t - 2c_1 e^{-\frac{1}{2}t}\sin 2t - \frac{1}{2}c_2 e^{-\frac{1}{2}t}\sin 2t + 2c_2 e^{-\frac{1}{2}t}\cos 2t$$

$$\dot{x}(0) = -\frac{1}{2}c_1 + 0 + 0 + 2c_2 \overset{\text{want}}{=} 2$$

$$\boxed{c_1 = 0} \qquad \boxed{c_2 = 1}$$

$$x(t) = e^{-\frac{1}{2}t}\sin 2t$$

$$x(2) = e^{-\frac{1}{2}\cdot 2}\sin 2\cdot 4 = e^{-1}\sin 4$$

A. ✓

18. A particular solution, y_p, of

$$y'' - 4y' + 3y = 2t + e^t$$

is?

A. $-\frac{1}{2}te^t + \frac{1}{3}t + \frac{1}{2}$ B. $-\frac{1}{2}te^t + \frac{1}{2}t + \frac{1}{2}$ C. $-\frac{1}{2}e^t + \frac{1}{3}t + \frac{1}{2}$ D. $t^2 + e^t$ E. $-\frac{1}{2}te^t + \frac{2}{3}t + \frac{8}{9}$

$r^2 - 4r + 3 = 0$

$(r-3)(r-1) = 0$

$r = 1, 3$

$y_c = c_1 e^t + c_2 e^{3t}$

$y_p = At + B + t[Ce^t]$

$y_p' = A + Ce^t + Cte^t$

$y_p'' = 2Ce^t + Cte^t$

want

$$\underbrace{(2Ce^t + Cte^t)}_{y_p''} - 4\underbrace{(A + Ce^t + Cte^t)}_{y_p'} + 3\underbrace{(At + B + Cte^t)}_{y_p} = 2t + e^t$$

$$\underbrace{-2C}_{=1}e^t + \underbrace{(3A)}_{=2}t + \underbrace{(-4A + 3B)}_{=0} = 2t + e^t$$

19. Determine the appropriate form for a particular solution $y_p(x)$ to the third-order differential equation

$$y^{(3)} + y'' - y' - y = \cos x + xe^{-x}.$$

A. $A\cos x + B\sin x + x^2(Cx+D)e^{-x}$ B. $A\cos x + x(Bx+C)e^{-x}$ C. $x^2(A\cos x + B\sin x) + (Cx+D)e^{-x}$
D. $A\cos x + Bxe^{-x}$ E. $A\cos x + B\sin x + (Cx+D)e^{-x}$

$r^3 + r^2 - r - 1 = 0 \leftarrow r=1$ is a root.

$$\begin{array}{r|l} & r^2+2r+1 \\ \hline r-1 & r^3+r^2-r-1 \\ & r^3-r^2 \\ \hline & 2r^2-r-1 \\ & 2r^2-2r \\ \hline & r-1 \\ & r-1 \\ \hline & 0 \end{array}$$

$(r-1)\underbrace{\left[r^2+2r+1\right]}_{(r+1)^2} = 0$

$r = 1, -1, -1$

$y_c = c_1 e^x + c_2 e^{-x} + c_3 x e^{-x}$

$y_p = A\cos x + B\sin x + \underbrace{x^2\left[(Cx+D)e^{-x}\right]}_{= Cx^3e^{-x} + Dx^2e^{-x}}$

safe! Do not solve homog.

x^2 is the smallest power that clears this danger

20. If $y'' + 5y' + 6y = 24e^t$, $y(0) = 0$, $y'(0) = 0$, then $y(1) = ?$

A. $e - e^{-2} + 6e^{-3}$ B. $2e - 8e^{-2} + 6e^{-3}$ C. $e - 8e^{-2} + 6e^{-3}$ D. $e + 8e^{-2} + e^{-3}$ E. 0

$r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0$

$r = -2, -3$

$y_p = Ae^t$ Get $A = 2$.

$y_c = c_1 e^{-2t} + c_2 e^{-3t}$

Gen^l Sol^n: $\begin{cases} y = c_1 e^{-2t} + c_2 e^{-3t} + 2e^t \\ y' = -2c_1 e^{-2t} - 3c_2 e^{-3t} + 2e^t \end{cases}$

want

$\begin{cases} y(0) = c_1 + c_2 + 2 = 0 \\ y'(0) = -2c_1 - 3c_2 + 2 = 0 \end{cases}$

Get c_1, c_2.

$y(1) = c_1 e^{-2} + c_2 e^{-3} + 2e$

21. The differential equation

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = 0$$

has solutions $y_1(t) = t$ and $y_2(t) = t^2$. If

$$1 \cdot y'' - \frac{2}{t}y' + \frac{2}{t^2}y = 2; \quad y(1) = 0, \; y'(1) = 0$$

then $y(2) = ?$

A. $8\ln 2 - 4$ B. 0 C. -6 D. $8\ln 2 + 4$ E. $8\ln 2$

$y_p = u_1 y_1 + u_2 y_2$ where $\boxed{u_1' = \frac{-y_2 F}{W} \qquad u_2' = \frac{y_1 F}{W}}$

standard form! $F(t) = 2$

$$W = \det \begin{bmatrix} t & t^2 \\ 1 & 2t \end{bmatrix} = 2t^2 - t^2 = t^2$$

$$u_1' = \frac{-(t^2)(2)}{(t^2)} = -2, \quad \text{So} \quad u_1 = \int -2\,dt = -2t$$

$$u_2' = \frac{(t)(2)}{t^2} = \frac{2}{t}, \quad \text{So} \quad u_2 = \int \frac{2}{t}\,dt = 2\ln t$$

$$y_p = (-2t)(t) + (2\ln t)(t^2)$$

$$= \underbrace{-2t^2}_{\text{a sol}^n \text{ to } L[y]=0!} + 2t^2 \ln t$$

$$2 = L[y_p] = \underbrace{L[-2t^2]}_{=0} + L[\underbrace{2t^2 \ln t}_{\text{better } y_p!}]$$

Gen^l Sol^n: $y = c_1 t + c_2 t^2 + 2t^2 \ln t$, etc.

22. A spring-mass system is governed by the initial value problem

$$x'' + 4x' + 4x = 4\cos\omega t$$
$$x(0) = 9, \quad x'(0) = -2.$$

For what value(s) of ω will resonance occur?

A. 0 B. 2 C. 4 D. no value of ω E. $2 < \omega <$ inf

$r^2 + 4r + 4 = 0$
$(r+2)^2 = 0$
$r = -2, -2$
$y_c = c_1 e^{-2t} + c_2 t e^{-2t}$

$y_p = A\cos\omega t + B\sin\omega t$
$= a\cos(\omega t - \varphi)$

Resonace cannot happen!

23. Rewrite the second order equation

$$2u'' + 3u' + ku = \cos 2t$$

as a system of first order equations.

A. $\begin{cases} x' = y \\ y' = \frac{1}{2}(-3x - ky + \cos 2t) \end{cases}$ B. $\begin{cases} x' = x \\ y' = \frac{1}{2}(-3y - kx + \cos 2t) \end{cases}$ C. $\begin{cases} x' = y \\ y' = \frac{1}{2}(-3y - kx + \cos 2t) \end{cases}$

D. $\begin{cases} x' = y \\ y' = 2y + kx + \cos 2t \end{cases}$ E. $\begin{cases} x' = 2y + kx + \cos 2t \\ y' = x \end{cases}$

$$\begin{cases} x = u \\ y = u' \end{cases} \qquad x_1 = u, \quad x_2 = u'$$

$$\begin{cases} \dfrac{dx_1}{dt} = u' = x_2 \\ \dfrac{dx_2}{dt} = u'' = -\frac{3}{2}u' - \frac{k}{2}u + \frac{1}{2}\cos 2t \end{cases}$$

from 2nd order ODE

$$= -\frac{3}{2}x_2 - \frac{k}{2}x_1 + \frac{1}{2}\cos 2t$$

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = -\frac{k}{2}x_1 - \frac{3}{2}x_2 + \frac{1}{2}\cos 2t \end{cases}$$

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{2} & -\frac{3}{2} \end{bmatrix}\vec{x} + \begin{pmatrix} 0 \\ \frac{1}{2}\cos 2t \end{pmatrix}$$

24. The solution of

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

is?

A. $2e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{-t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ B. $2e^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ C. $e^{3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ D. $3e^{3t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} - e^{-t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

E. $3e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{-t}\begin{bmatrix} 0 \\ -4 \end{bmatrix}$

$$\det\begin{bmatrix} 1-r & 1 \\ 4 & 1-r \end{bmatrix} = (1-r)^2 - 4 = 0$$
$$= r^2 - 2r - 3 = 0$$
$$(r-3)(r+1) = 0 \qquad r = -1, 3$$

For $r=-1$: $\quad (A - r\mathbb{I})\vec{a} = 0$, $r=-1$

$$\left[\begin{array}{cc|c} 1-(-1) & 1 & 0 \\ 4 & -1-(-1) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & 1 & 0 \\ \cancel{4} & \cancel{2} & \cancel{0} \end{array}\right] \qquad \vec{a} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

For $r=3$

$$\left[\begin{array}{cc|c} 1-3 & 1 & 0 \\ 4 & 1-3 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} -2 & 1 & 0 \\ \cancel{4} & \cancel{-2} & \cancel{0} \end{array}\right]$$

$$\vec{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Gen^l sol^n $\vec{x} = c_1\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{-t} + c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{3t}$

want

$$\vec{x}(0) = c_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$c_1 = \frac{\det\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}}{\det\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}} = \frac{4}{4} = 1 \qquad c_2 = \frac{\det\begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix}}{\det\begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}} = \frac{8}{4} = 2$$

25. Solve

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

A. $\mathbf{x}(t) = 2e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} - e^t \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$ B. $\mathbf{x}(t) = 2e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + e^t \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$ C. $\mathbf{x}(t) = 2e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} - e^t \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$

D. $\mathbf{x}(t) = e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} - e^t \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$ E. $\mathbf{x}(t) = e^t \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} - e^t \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$

$$\det \begin{pmatrix} 1-r & 1 \\ -1 & 1-r \end{pmatrix} = (1-r)^2 + 1 = 0$$

$$(1-r) = \pm i$$

$$r = 1 \pm i$$

For $r = 1+i$: $(\underset{r=1+i}{A - rI})\vec{a} = \vec{0}$

$$\left[\begin{array}{cc|c} 1-(1+i) & 1 & 0 \\ -1 & 1-(1+i) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} -i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{a} = \begin{pmatrix} 1 \\ -(-i) \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ i \end{pmatrix}(e^t \cos t + i e^t \sin t)$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\underbrace{\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t \cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t \sin t\right]}_{\vec{x}_1} + i \underbrace{\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t \sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t \cos t\right]}_{\vec{x}_2}$$

Gen'l Sol'n $\quad c_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^t + c_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} e^t$

26. Solve the initial value problem

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix}1\\1\end{bmatrix}, \quad \text{where } A = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}.$$

$\det\begin{pmatrix}1-r & 1\\0 & 1-r\end{pmatrix} = (1-r)^2 = 0$

$r = 1, 1$ repeated!

A. $e^t\begin{bmatrix}1\\1\end{bmatrix} - 2te^t\begin{bmatrix}1\\0\end{bmatrix}$. B. $e^t\begin{bmatrix}1\\1\end{bmatrix} + te^t\begin{bmatrix}1\\0\end{bmatrix}$. C. $e^t\begin{bmatrix}1\\1\end{bmatrix} + 2te^t\begin{bmatrix}1\\0\end{bmatrix}$. D. $e^t\begin{bmatrix}0\\1\end{bmatrix} + 2te^t\begin{bmatrix}1\\0\end{bmatrix}$. E. $e^t\begin{bmatrix}1\\1\end{bmatrix} - 2te^t\begin{bmatrix}1\\0\end{bmatrix}$.

For $r=1$ $\quad \left[\begin{array}{cc|c}1-1 & 1 & 0\\0 & 1-r & 0\end{array}\right] \sim \left[\begin{array}{cc|c}0 & 1 & 0\\0 & 0 & 0\end{array}\right] \quad \vec{a} = \begin{pmatrix}1\\0\end{pmatrix}$

Get $\vec{x}_1 = \begin{pmatrix}1\\0\end{pmatrix}e^t$

Second sol^n $\quad \vec{x}_2 = \vec{a}te^t + \vec{b}e^t \quad$ where $(A - \overset{r=1}{r}I)\vec{b} = \vec{a}$

$b_2 = 1$

$b_1 =$ anything $\quad \longrightarrow \left[\begin{array}{cc|c}0 & 1 & 1\\0 & 0 & 0\end{array}\right]$

Take $\vec{b} = \begin{pmatrix}0\\1\end{pmatrix}$. $\quad$ Get $\vec{x}_2 = \begin{pmatrix}1\\0\end{pmatrix}te^t + \begin{pmatrix}0\\1\end{pmatrix}e^t = \begin{pmatrix}t\\1\end{pmatrix}e^t$

Gen^l sol^n $\quad \vec{x} = c_1\begin{pmatrix}1\\0\end{pmatrix}e^t + c_2\begin{pmatrix}t\\1\end{pmatrix}e^t$

Want $\quad \vec{x}(0) = c_1\begin{pmatrix}1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \quad$ Get $c_1 = 1$, $c_2 = 1$.

Sol^n $\quad \vec{x}(t) = \begin{pmatrix}1\\0\end{pmatrix}e^t + \begin{pmatrix}t\\1\end{pmatrix}e^t$

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27. What values of the parameter α in the system below make the origin a saddle point in the phase plane:

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ \alpha & 2 \end{bmatrix} \mathbf{x}$$

A. $\alpha > 2$ B. $\alpha > -\frac{1}{4}$ C. $\alpha < -\frac{1}{4}$ D. $2 > \alpha > -\frac{1}{4}$ E. $\alpha < -2$

$$\text{Det}\begin{pmatrix} 1-r & 1 \\ \alpha & 2-r \end{pmatrix} = 0 \qquad (1-r)(2-r) - \alpha = 0$$

$$r^2 - 3r + (2-\alpha) = 0$$

Roots $r = \dfrac{3 \pm \sqrt{9 - 4(2-\alpha)}}{2} = \dfrac{3}{2} \pm \dfrac{\sqrt{1+4\alpha}}{2}$

Saddle pt: r = real, opposite sign.

So we need $\boxed{1 + 4\alpha > 0}$ for real

$4\alpha > -1$

$\boxed{\alpha > -\frac{1}{4}}$

$\frac{3}{2} + \frac{\sqrt{1+4\alpha}}{2}$ is positive.

We need $\frac{3}{2} - \frac{\sqrt{1+4\alpha}}{2}$ to be negative, i.e.

$$\frac{3}{2} < \frac{\sqrt{1+4\alpha}}{2}$$

$$\frac{9}{4} < \frac{1+4\alpha}{4}$$

$$9 < 1 + 4\alpha$$

$$2 < \alpha$$

So we need $\alpha > -\frac{1}{4}$ and $\alpha > 2$, i.e.,

$$\boxed{\alpha > 2}$$