Lecture 31 Laplace transform 7.1 Today: MyLab HW 30

Monday: MyLab HW 31, and Gradescope HW 29, 30, 31

Fourier transform: $\hat{f}(s) = \int_{-\infty}^{\infty} f(t)e^{-ist}\,dt$ ← with maybe a $\frac{1}{2\pi}$ in front or inside $e^{-i2\pi st}$ or not.

Laplace transform: Takes function $f(t)$ on $t \geq 0$ and sends it to new function $F(s)$ for $s >$ (something)

$$\mathcal{L}[f](s) = \int_0^{\infty} f(t)e^{-st}\,dt = F(s)$$

← Laplace transform of $f(t)$

$\mathcal{L}[f] = F(s)$

Huge facts

1) $\mathcal{L}[f'(t)] = sF(s) - f(0)$

$$\mathcal{L}[f''(t)] = \mathcal{L}[(f')'] = s\,\underbrace{\mathcal{L}[f']}_{= sF(s)-f(0)} - f'(0)$$

$$s\cdot(sF(s) - f(0))$$

$$\mathcal{L}[f''] = s^2F(s) - sf(0) - f'(0)$$

2) $\mathcal{L}$ is a linear operator: $\mathcal{L}[c_1f_1 + c_2f_2] = c_1F_1 + c_2F_2$

Easy because $\int$ is linear.

3) If $\mathcal{L}[f_1] = \mathcal{L}[f_2]$ for $s > M$ (for some M), then f_1 and f_2 must be the same!

Consequently, $\mathcal{L}^{-1}$ makes sense and we can "undo" the Lap transf.

Master plan $\quad Ly'' + Ry' + \frac{1}{C}y = e(t)$, $\quad \begin{cases} y(0) = y_0 \\ y'(0) = y_0 \end{cases}$ IVP

(RLC circuit: $e(t) = v'(t)$, $y = I$)

Hit eqn with $\mathcal{L}$:

$$L\left(s^2 Y - s y(0) - y'(0)\right) + R\left(sY - y(0)\right) + \frac{1}{C}Y = E(s)$$

$$\left(Ls^2 + Rs + \frac{1}{C}\right)Y - sLy_0 - Ly_0' - Ry_0 = E(s)$$

Solve for Y:

$$Y = \frac{(sL+R)y_0 + Ly_0'}{Ls^2 + Rs + \frac{1}{C}} + \frac{E(s)}{Ls^2 + Rs + \frac{1}{C}}$$

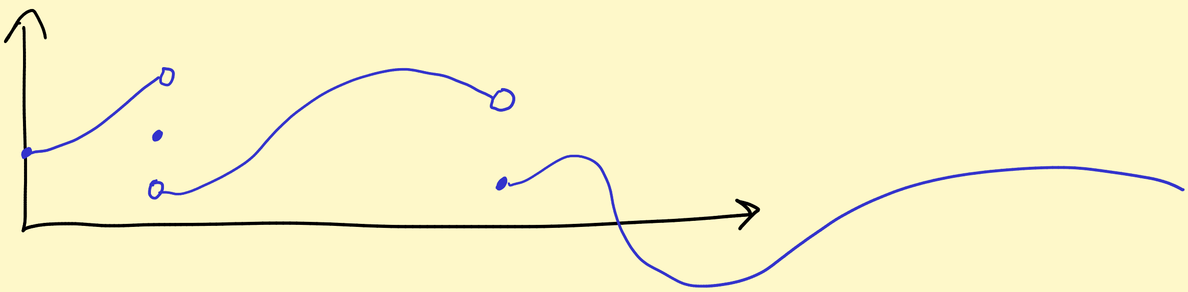
Undo $\mathcal{L}$. Get sol^n $\quad y = \mathcal{L}^{-1}(\text{RHS})$

First term yields a sol^n to homog eqn.
Second term yields a particular sol^n.

Exciting thing: $\mathcal{L}$: ODE's $\longrightarrow$ algebra eqn

solve to get Y.
Then undo $\mathcal{L}$
to get sol^n $y = \mathcal{L}^{-1}[Y]$

<u>Requirements</u> 1) Need $f(t)$ to be piecewise continuous



2) Need f to be "of exponential type", meaning there are constants $M>0$ and $K>0$ such that

$$|f(t)| \le M e^{Kt} \quad \text{for } t \ge 0.$$

<u>EX</u> $t^2 e^{3t} \cos 2t$

Hmmm. Fresh calc.: $\frac{t^2}{e^t} \to 0$ as $t \to \infty$

So $\frac{t^2}{e^t} < 1$ for $t > A$ for some A.

So $t^2 < e^t$ for $t > A$

$$\left| t^2 e^{3t} \cos 2t \right| \le \underbrace{|t^2|}_{t^2 < e^t} \, |e^{3t}| \, \underbrace{|\cos 2t|}_{\le 1} \quad \text{when } t > A.$$

$$\le \underbrace{e^t \cdot e^{3t} \cdot 1}_{e^{4t}} \quad \text{when } t > A$$

Hmmm. $\left| t^2 e^{3t} \cos 2t \right|$ is <u>bounded</u> on $[0, A]$, say by M.

Get $\left| t^2 e^{3t} \cos 2t \right| \le \mathrm{Max}(M,1) \cdot e^{4t}$ for all $t \geq 0$.

EX $e^{(e^t)}$ is not of expo type.

Start our "table of Lap transf"

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a} \quad \text{for} \quad s > a.$$

$$\mathcal{L}\left[e^{at}\right] = \int_0^\infty e^{at} e^{-st}\, dt = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \lim_{B\to\infty} \int_0^B e^{-(s-a)t}\, dt$$

$$= \lim_{B\to\infty} \left[\frac{1}{-(s-a)} e^{-(s-a)t}\right]_0^B$$

$$= -\frac{1}{s-a} \lim_{B\to\infty} \left[e^{-(s-a)B} - 1\right]$$

e^0

must have $s-a > 0$ for this to have a limit as $B \to \infty$. $\lim = 0$

So, for $s > a$, get $\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}$ ✓

$$\mathcal{L}[1] = \int_0^\infty 1 \cdot e^{-st}\, dt = \lim_{B\to\infty} \int_0^B e^{-st}\, dt$$

$$= \lim_{B \to \infty} \left[-\frac{1}{s} e^{-st} \right]_0^B$$

$$= \lim_{B \to \infty} \left[-\frac{1}{s} e^{-sB} - \left(-\frac{1}{s}\right) e^{0} \right]$$

need $s > 0$: $e^{-sB} \to 0$ as $B \to \infty$.

$$= \frac{1}{s} \quad \text{when } s > 0.$$

$$\mathcal{L}[t^2] = \int_0^\infty \underbrace{t^2}_{u} \; \underbrace{e^{-st} \, dt}_{dv}$$

$u = t^2$

$du = 2t \, dt$

$v = \int e^{-st} dt = -\frac{1}{s} e^{-st}$

$$= uv - \int v \, du$$

$$= -\frac{t^2}{s} e^{-st} \Big|_0^\infty - \underbrace{\int_0^\infty \left(-\frac{1}{s} e^{-st}\right)(2t \, dt)}$$

Blast! Need to do int by parts again!

$u = t \quad v = e^{-st}$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \quad \text{for } s > 0$$

$$\mathcal{L}[t^2] = \frac{2!}{s^3} \qquad \mathcal{L}[t] = \frac{1}{s^2}$$