Lecture 32 Solving ODEs with Laplace transform 7.2

MyLab HW 31 and Gradescope HW 29, 30, 31 due ~~today~~ Monday

Table of Laplace transforms p. 454

$f(t)$	$F(s)$	
e^{at}	$\frac{1}{s-a}$	$(s>a)$
$\sin bt$	$\frac{b}{s^2+b^2}$	$(s>0)$
$\cos bt$	$\frac{s}{s^2+b^2}$	$(s>0)$
1	$\frac{1}{s}$	$(s>0)$
t	$\frac{1}{s^2}$	$(s>0)$
t^n	$\frac{n!}{s^{n+1}}$	$(s>0)$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$	$(s>a)$
$e^{at}\cos bt$	$\frac{(s-a)}{(s-a)^2+b^2}$	
$f'(t)$	$sF(s)-f(0)$	$\mathcal{L}[f(t)]=F(s)$
$f''(t)$	$s^2F(s)-f(0)s-f'(0)$	
$f^{(n)}(t)$	$s^nF(s)-f(0)s^{n-1}-f'(0)s^{n-2}-\cdots-f^{(n-1)}(0)$	

$\int_0^t f(u)\,du$	$\frac{F(s)}{s}$

EX $\mathcal{L}[\cosh bt] = \mathcal{L}\left[\frac{1}{2}\left(e^{bt} + e^{-bt}\right)\right]$

$= \frac{1}{2}\mathcal{L}[e^{bt}] + \frac{1}{2}\mathcal{L}[e^{-bt}]$ ← $a = -b$

$= \frac{1}{2}\frac{1}{s-b} + \frac{1}{2}\frac{1}{s-(-b)}$

$s > b$ $\qquad$ $s > (-b)$

Need $s > b$ for both to make sense

$\cosh bt$	$\frac{s}{s^2 - b^2}$	$(s > b)$
$\sinh bt$	$\frac{b}{s^2 - b^2}$	$\sinh bt = \frac{1}{2}\left(e^{bt} - e^{-bt}\right)$

Cool thing $\quad \mathcal{L}[\cos bt] = \mathcal{L}\left[\frac{1}{2}\left(e^{ibt} + e^{-ibt}\right)\right]$

$= \frac{1}{2}\mathcal{L}[e^{ibt}] + \frac{1}{2}\mathcal{L}[e^{-ibt}]$

$= \frac{1}{2}\frac{1}{s-ibt} + \frac{1}{2}\frac{1}{s-(-ib)}$ $\qquad (s > 0)$

$\cos bt$	$\frac{s}{s^2+b^2}$	
$\sin bt$	$\frac{b}{s^2+b^2}$	← $\sin bt = \frac{1}{2}\left(e^{ibt} - e^{-ibt}\right)$

<u>or</u> $\mathcal{L}[\cos bt] = \int_0^\infty \underset{\uparrow \atop u}{\cos bt}\ \underbrace{e^{-st}\, dt}_{dv}$

get a similarly hard $\int$ with Sin in place of Cos

Aha! Do it again! Get a $\cos bt \cdot e^{-st}$ $\int$ again with $c \neq 1$ in front of it. Solve for the $\int$.

<u>Reduction formula</u> $\mathcal{L}[t^n] = \frac{n}{s}\mathcal{L}[t^{n-1}]$

via integration by parts. Repeat n-times

$$\mathcal{L}[t^n] = \overbrace{\frac{n}{s} \cdot \frac{(n-1)}{s} \cdot \frac{(n-2)}{s} \cdots \frac{1}{s}}^{n \text{ things}} \cdot \underbrace{\mathcal{L}[1]}_{= \frac{1}{s}} = \frac{n!}{s^{n+1}}$$

$\mathcal{L}[t]$

<u>EX</u> $y'' + 5y' + 6y = 0$ $\begin{cases} y(0) = 3 \\ y'(0) = 5 \end{cases}$

$$(s^2 Y - \underset{\uparrow \atop y(0)}{3}s - \underset{\uparrow \atop y'(0)}{5}) + 5(sY - \underset{\uparrow \atop y'(0)}{3}) + 6Y = 0$$

$$(s^2 + 5s + 6)Y = 3s + 20$$

$$Y = \frac{3s + 20}{s^2 + 5s + 6}$$

Need $\mathcal{L}^{-1}$ of this!   Key for this one: Partial fractions!

$$\bar{Y} = \frac{3s+20}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

Multiply by denom: $3s+20 = A(s+3) + B(s+2)$

Collect coeff: $3s + 20 = (A+B)s + (3A+2B)$

Equate coeff: $\begin{cases} A+B = 3 \\ 3A+2B = 20 \end{cases}$

Solve:

$$A = \frac{\det\begin{bmatrix} 3 & 1 \\ 20 & 2 \end{bmatrix}}{\det\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}} = \frac{(-14)}{(-1)} = 14$$

$$B = \frac{\det\begin{bmatrix} 1 & 3 \\ 3 & 20 \end{bmatrix}}{\det\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}} = \frac{11}{(-1)} = -11$$

So $\bar{Y} = \dfrac{14}{s+2} - \dfrac{11}{s+3}$ $\mathcal{L}[e^{at}] = \dfrac{1}{s-a}$

$+2 = -a$

$a = -2$

and $y = 14e^{-2t} - 11e^{-3t}$

Overdamped case!

Dealing with $\dfrac{ds+e}{as^2+bs+c} = \bar{Y}(s)$

1) Case Real, distinct roots

$$as^2+bsc = a(s-r_1)(s-r_2)$$

$$Y(s) = \frac{A}{s-r_1} + \frac{B}{s-r_2} \qquad \text{Easy!}$$

2) $as^2+bs+c = a(s-r)^2 \leftarrow$ r real, repeated root

Partial fractions! $\dfrac{ds+e}{a(s-r)^2} = \dfrac{A}{(s-r)^2} + \dfrac{B}{s-r}$

Hmmm. $A\mathcal{L}[te^{-rt}]$? $B\mathcal{L}[e^{-rt}]$

(Critically damped case) (Yes!)

3) as^2+bs+c has complex roots (in conjugate pairs)

<u>Step 1</u> <u>Complete the square</u> in the denom

Completing the square: as^2+bs+c

$$a\left(s^2+\underset{\substack{\uparrow\\B}}{\frac{b}{a}}s+\underset{\substack{\uparrow\\C}}{\frac{c}{a}}\right)$$

$$s^2+Bs+C = \left(s+\frac{B}{2}\right)^2 + \left(C-\frac{B^2}{4}\right)$$

Half the B

clean up the extra const

EX $s^2 + 4s + 13 = \left(s + \frac{4}{2}\right)^2 + (13 - 4)$

$$= (s+2)^2 + 9$$

Hmmm

$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos bt$	$\frac{(s-a)}{(s-a)^2+b^2}$

EX $\frac{3s+4}{s^2+4s+13} = \frac{3s+4}{(s+2)^2+3^2}$

Favorite trick: Wish I had $s+2$ on top!
Give it to yourself!

$$s = [(s+2) - 2]$$

$$\frac{3s+4}{(s+2)^2+3^2} = \frac{3[(s+2)-2]+4}{(s+2)^2+3^2}$$

wish!

$$= 3 \cdot \frac{(s+2)}{(s+2)^2+3^2} - \frac{2}{3} \cdot \frac{1 \cdot 3}{(s+2)^2+3^2}$$

$a = -2, \quad b = 3$

$$= \mathcal{L}\left[3e^{-2t}\cos 3t - \frac{2}{3}e^{-2t}\sin 3t\right]$$

Underdamped case!