Lecture 33 s-shifting and t-shifting rules 7.3

MyLab HW 32 tonight. Friday: MyLab HW 33 and GS HW 32,33

Fun thing: $\int e^{at}\sin bt = \underline{Ae^{at}\cos bt + Be^{at}\sin bt}$

Use undetermined coeff to get a particular sol^n to

$\frac{dy}{dt} = e^{at}\sin bt$ (Note: Homogeneous sol^n to $\frac{dy}{dx}=0$ is $c\cdot 1$)

Why $\mathcal{L}[f'(t)] = sF(s) - y(0)$

$$\mathcal{L}[f'(t)] = \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{f'(t)\,dt}_{dv} \qquad \begin{cases} du = -se^{-st}\,dt \\ v = f(t) \end{cases}$$

$$= \lim_{B\to\infty}\left(\underbrace{e^{-st}}_{u}\underbrace{f(t)}_{v}\Big|_0^B - \int_0^B \underbrace{f(t)}_{v}\underbrace{\left[-se^{-st}\right]dt}_{du} \right)$$

Note: f of exponential type: $|f(t)| \le Me^{Kt}$

So $\lim_{t\to\infty} e^{-st}f(t) = 0$ if $(s > K)$ ← typical thing in $\mathcal{L}$ table

$$= \left(0 - 1\cdot f(0)\right) + s\underbrace{\int_0^\infty f(t)e^{-st}\,dt}_{F(s)} \quad \checkmark$$

What about antiderivative formula?

$\int_0^t f(\tau)\,d\tau = g(t)$. Note: $g(0)=0$ and $g'(t) = f(t)$.

Write $\mathcal{L}[g] = G$.

$$\mathcal{L}[g'(t)] = s\,G(s) - \overbrace{g(0)}^{0=\int_0^0}$$

$$\mathcal{L}[f(t)] = s\,\mathcal{L}\Big[\underbrace{\int_0^t f(\tau)\,d\tau}_{g(t)}\Big] - 0$$

$$\mathcal{L}\left[\int_0^t f(\tau)\,d\tau\right] = \frac{F(s)}{s} \quad \checkmark$$

Important shifting rules $\qquad \mathcal{L}[f(t)] = F(s)$

1) s-shift rule:

$$\mathcal{L}[e^{at} f(t)] = \underbrace{F(s-a)}$$

F is "shifted" by a (replace s by s−a)

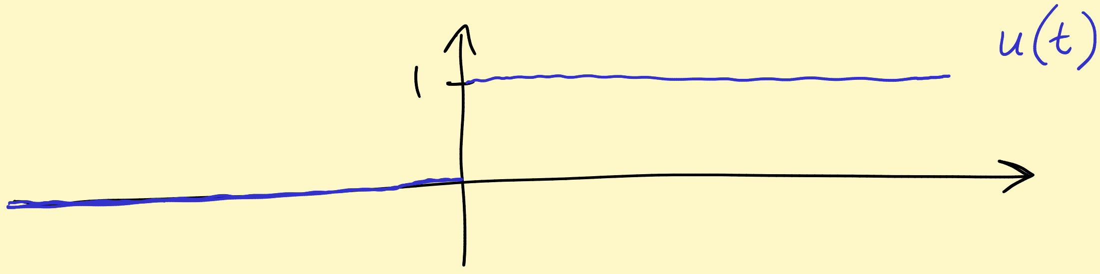
2) t-shift rule:

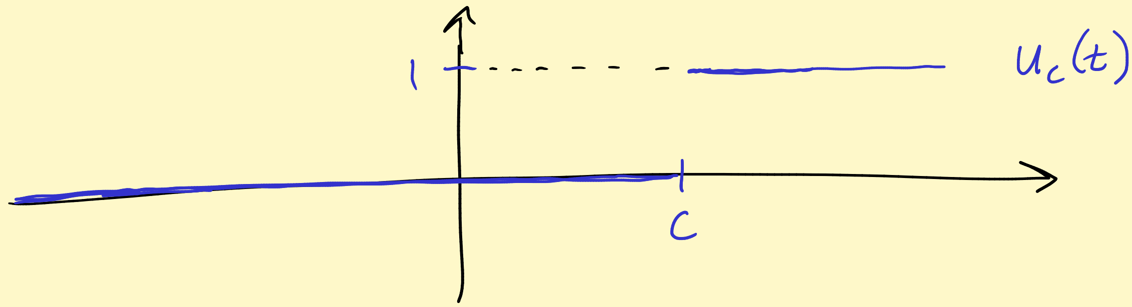
$$\mathcal{L}[u_c(t)\, f(t-c)] = e^{-cs} F(s)$$

f is shifted by c (replace t by t−c)

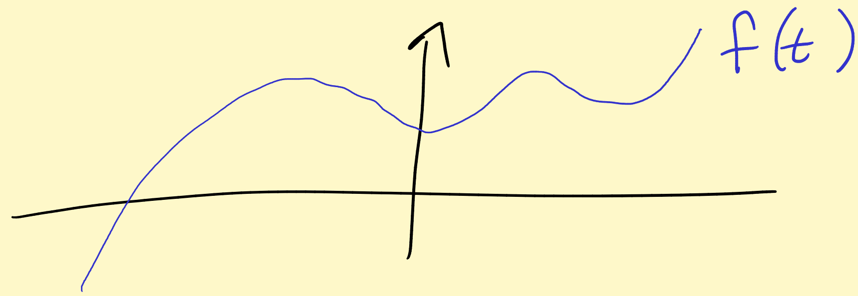
Need this because $f(t)=0$ for $t<0$.

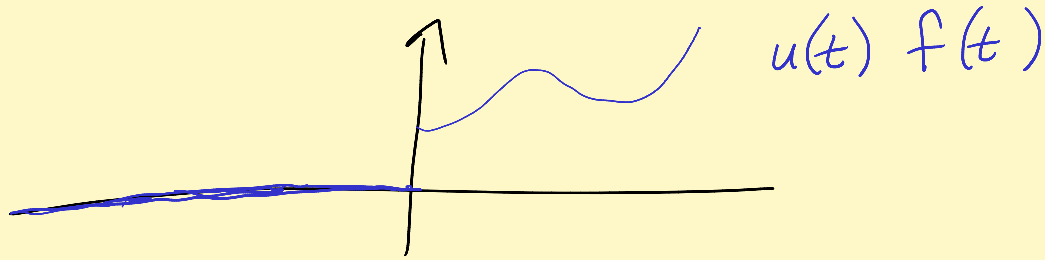
Heaviside function (or "unit step function")

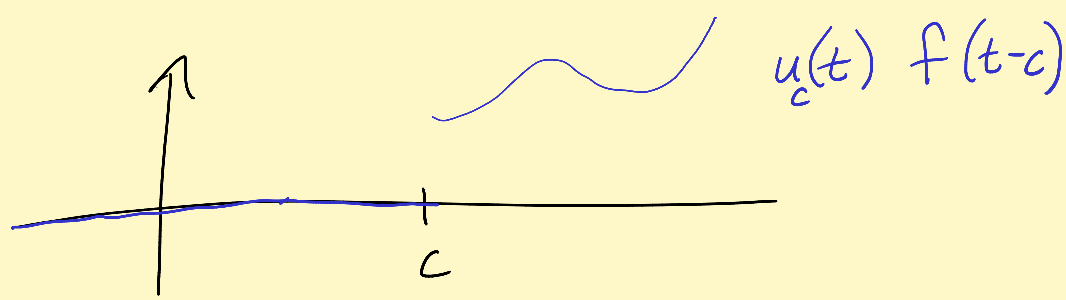




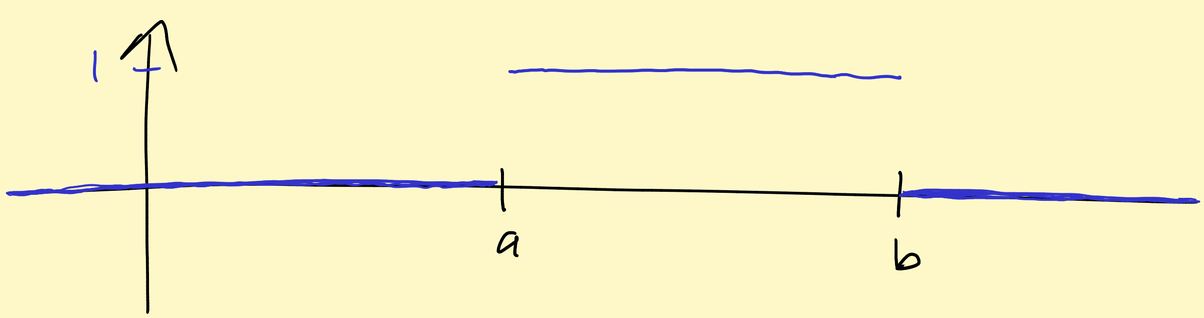
Def^n $\quad u_c(t) = u(t-c)$ $\leftarrow$ turns on at time c



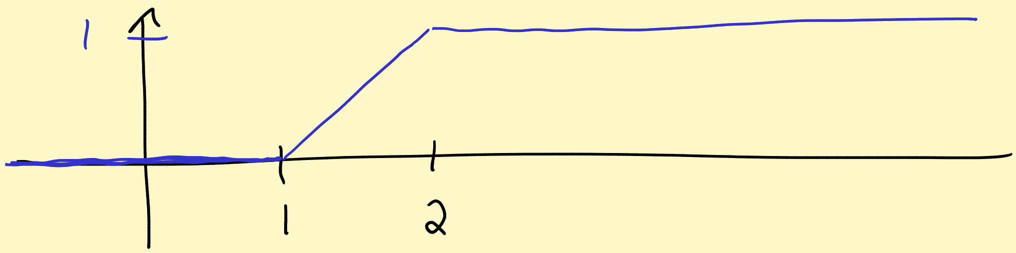




Great tool to write formulas for piecewise defined fcns.



$u_a(t) - u_b(t)$ $\qquad$ Turns on at $t=a$ and off at $t=b$.



$$\left[u_1(t)-u_2(t)\right](t-1) + u_2(t)$$

Hmmm. What is $\mathcal{L}$ of this? $\mathcal{L}\left[u_c(t)f(t-c)\right] = e^{-cs}F(s)$

$$u_1(t)\underbrace{(t-1)}_{f(t-1)} - u_2(t)\underbrace{(t-1)}_{\text{wish this were } t-2} + u_2(t)\cdot 1$$

$f(t) = t$

give it to yourself!

$h(t) = 1$

$h(t-2) = 1$ too

$$u_2(t)\left[t-2+2-1\right]$$

$$\underbrace{(t-2)+1}_{g(t-2)}$$

$$g(t) = t+1$$

$$\mathcal{L} = e^{-1\cdot s}\underbrace{\frac{1}{s^2}}_{\mathcal{L}[t]} - e^{-2s}\underbrace{\left(\frac{1}{s^2}+\frac{1}{s}\right)}_{\mathcal{L}[t+1]} + e^{-2s}\cdot\underbrace{\frac{1}{s}}_{\mathcal{L}[1]}$$

Table	
$u_c(t)$	$\dfrac{e^{-cs}}{s}$

$$\mathcal{L}\left[u_c(t)\right] = \int_0^\infty u_c(t)\,e^{-st}\,dt = \int_c^\infty e^{-st}\,dt \quad \text{easy.}$$

Why s-shift rule $\quad \mathcal{L}[e^{at} f(t)]$

$$= \int_0^\infty e^{at} f(t) e^{-st}\, dt = \int_0^\infty f(t) e^{-(s-a)t}\, dt$$

$$= F(s-a) \checkmark$$

Why t-shift rule

$$\mathcal{L}[u_c(t) f(t-c)] = \int_0^\infty u_c(t) f(t-c) e^{-st}\, dt$$

$$= \int_{t=c}^\infty f(\underbrace{t-c}_{\tilde{\tau}}) e^{-st}\, dt$$

$\tilde{\tau} = t - c$

$d\tilde{\tau} = dt$

$t = \tilde{\tau} + c$

When $t=c$, $\tilde{\tau}=0$.

As $t \to \infty$, $\tilde{\tau} \to \infty$.

$$= \int_{\tilde{\tau}=0}^\infty f(\tilde{\tau}) e^{-s(\tilde{\tau}+c)}\, d\tilde{\tau}$$

$$= e^{-sc} \int_{\tilde{\tau}=0}^\infty f(\tilde{\tau}) e^{-s\tilde{\tau}}\, d\tilde{\tau}$$

change variable of integration back to t!

$$= e^{-cs} F(s) \checkmark$$

s-shift example: $\quad \mathcal{L}[e^{at} \cos bt] = \dfrac{(s-a)}{(s-a)^2 + b^2}$

Important trick

$$t = (t-c) + c$$

EX $\mathcal{L}\left[u_3(t)\, t^2\right]$

Hmmm. $\mathcal{L}\left[u_3(t)\, \underbrace{(t-3)^2}\right] = e^{-3s}\dfrac{2}{s^3}$

$f(t-3) = (t-3)^2$

$f(t) = t^2$ $\qquad \mathcal{L}[t^2] = \dfrac{2}{s^3}$

$$\mathcal{L}\left[u_3(t)\, t^2\right] = \mathcal{L}\left[u_3(t)\, \left((t-3)+3\right)^2\right]$$

$$= \mathcal{L}\left[u_3(t)(t-3)^2 + 2\cdot 3\, u_3(t)(t-3) + 9\, u_3(t)\right]$$

$$= e^{-3s}\cdot\frac{2}{s^3} + 6\, e^{-3s}\cdot\frac{1}{s^2} + 9\,\frac{e^{-3s}}{s}$$

Partial fractions

$$\frac{s^{10} + 3s^5 + 1}{s^2 (s-3)^3 \underbrace{(s^2+2s+2)}^2 \underbrace{(s^2+9)}}$$

deg $10 < 11$ ✓

← deg $= 11$

$s = s-0$

(complex roots)

$$= \left(\frac{A}{s^2} + \frac{B}{s}\right) + \left(\frac{C}{(s-3)^3} + \frac{D}{(s-3)^2} + \frac{E}{(s-3)}\right)$$

$$\left(\frac{Fs+G}{(s^2+2s+2)^2} + \frac{Hs+I}{(s^2+2s+2)}\right) + \frac{Js+K}{s^2+9}$$