Lecture 34 Convolution 7.4

MyLab HW 33

GS HW 32, 33

$$\mathcal{L}[t \sin t] \stackrel{?}{=} \cancel{\frac{1}{s^2} \cdot \frac{1}{s^2+1}}$$

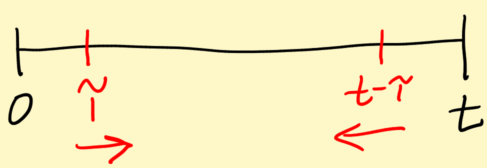
No!, No!, No!

$\mathcal{L}[t]$ $\quad$ $\mathcal{L}[\sin t]$

True formula $\quad \mathcal{L}[f * g] = F(s)\,G(s)$

where $\displaystyle (f*g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$

= "the convolution of f and g"



Discrete version $\quad \displaystyle \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n$

where $\displaystyle c_n = \sum_{k=0}^{n} a_k\, b_{n-k}$

Warning Today's convolution is for Laplace transform.

Convolution for Fourier transform: $\displaystyle \int_{-\infty}^{\infty} f(\tau)\,g(t-\tau)\,d\tau$.

EX Find $\mathcal{L}^{-1}\left[\underbrace{\frac{1}{(s^2+1)^2}}_{\left(\underbrace{\mathcal{L}[\sin t]}\right)^2}\right]$

$f(t) = \sin t$, $F(s) = \frac{1}{s^2+1}$

$$\mathcal{L}\left[(f*f)(t)\right] = F(s)\cdot F(s) = F(s)^2 \checkmark$$

Ans: $(f*f)(t) = \displaystyle\int_0^t (\sin \tilde{\tau})\left(\sin(t-\tilde{\tau})\right) d\tilde{\tau}$

Hmmm $\sin\alpha \sin\beta = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta)$

$\alpha = \tilde{\tau}$, $\beta = t - \tilde{\tau}$

$$\sin\tilde{\tau}\,\sin(t-\tilde{\tau}) = \frac{1}{2}\cos(2\tilde{\tau}-t) - \frac{1}{2}\cos t$$

So $(f*f)(t) = \displaystyle\int_{\tilde{\tau}=0}^t \frac{1}{2}\cos(\underbrace{2\tilde{\tau}-t}_{u}) - \frac{1}{2}\cos t \, d\tilde{\tau}$

$u = 2\tilde{\tau} - t$

$du = 2d\tilde{\tau}$

when $\tilde{\tau}=0$, $u=-t$

$\tilde{\tau}=t$, $u=t$

$$= \underbrace{\frac{1}{2}\int_{u=-t}^{t} \cos u \, \underbrace{\left(\tfrac{1}{2}du\right)}_{d\tilde{\tau}}}_{\frac{1}{4}\left[\sin u\right]_{u=-t}^{t}} - \left(\frac{1}{2}\cos t\right)\cdot \underset{\uparrow\ \int_0^t 1\cdot d\tilde{\tau}}{t}$$

$$= \frac{1}{4}\Big(\sin t - \underbrace{\sin(-t)}_{=-\sin t}\Big) - \frac{1}{2}\, t \cos t$$

$$= \frac{1}{2}\sin t - \frac{1}{2}\, t \cos t$$ ← Hmmm. Resonance!

Table	
$\frac{1}{2}\sin t - \frac{1}{2} t \cos t$	$\frac{1}{(s^2+1)^2}$

See book for $t\cos\omega t$, $t\sin\omega t$ formulas.

Another formula $\quad \mathcal{L}[t f(t)] = -F'(s)$

Cousin to $\mathcal{L}[f'(t)] = sF(s) - f(0)$

Consequence $\quad \mathcal{L}[t\cos t] = -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]$

Partial Fractions

1) $\frac{1}{s}$ $\qquad \mathcal{L}[1]$

2) $\frac{1}{s^2}$ $\qquad \mathcal{L}[t]$

3) $\frac{1}{s^n}$ $\qquad \mathcal{L}\left[\frac{1}{(n-1)!} t^{n-1}\right]$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

4) $\frac{1}{s-a}$ $\qquad \mathcal{L}[e^{at}]$

$\frac{1}{s+a}$ $\qquad \mathcal{L}[e^{-at}]$

5) $\frac{1}{(s-a)^2}$ $\qquad \mathcal{L}[t\,e^{at}]$

6) $\frac{As+B}{(s-a)^2+b^2}$ $\qquad$ Combo of $e^{at}\cos bt$, $e^{at}\sin bt$

7) $\frac{As+B}{s^2+b^2}$ $\qquad$ Combo of $\cos bt$, $\sin bt$

8) $\frac{As+B}{(s^2+b^2)^2}$ $\qquad$ Today's stuff

9) $\frac{As+B}{\left[(s-a)^2+b^2\right]^2}$ $\qquad$ Today's stuff plus s-shift

Exciting thing!

$$y'' + y = r(t) \qquad \begin{cases} y(0)=0 \\ y'(0)=0 \end{cases}$$

$$\left(s^2 Y(s) - s\cdot 0 - 0\right) + Y(s) = R(s)$$

$$Y(s) = \frac{1}{s^2+1}\cdot R(s)$$

Aha! $\mathcal{L}[\underbrace{(\sin * r)}_{y}] = \frac{1}{s^2+1} \cdot R(s)$

$$y(t) = \int_0^t \sin\tau \; \underbrace{r(t-\tau)}\, d\tau$$

great if r is from measurements!

Remark $f*g = g*f$ $\quad *$ is commutative!

Why $\mathcal{L}[f*g] = F(s)G(s) \overset{!}{=} G(s)F(s) = \mathcal{L}[g*f]$

$\mathcal{L}^{-1}$ is uniquely determined! So $f*g = g*f$ ✓

Get $\quad y(t) = \int_0^t r(\tau)\, \underbrace{\sin(t-\tau)}_{\text{kernel for the operator}}\, d\tau$

Integral operator: $r(t) \longmapsto y(t)$

Tricks for HW $\quad \mathcal{L}[t e^{at}] = ?$

Hmmm. $\quad f(t) = t e^{at} \longleftarrow f(0) = 0$

$$f'(t) = a\underline{t e^{at}} + e^{at}$$

$$\mathcal{L}[f'(t)] = a\,\mathcal{L}[f(t)] + \frac{1}{s-a}$$

$$s\mathcal{L}[f] - \underbrace{f(0)}_{0} = a\mathcal{L}[f] + \frac{1}{s-a}$$

Solve for $\mathcal{L}[f]$. Get $\mathcal{L}[te^{at}] = \frac{1}{(s-a)^2}$

Duh $\mathcal{L}[t\,g(t)] = -G'(s)$

$$\mathcal{L}[te^{at}] = -\frac{d}{ds}\left[\frac{1}{s-a}\right] = \frac{1}{(s-a)^2} \checkmark$$

Demo that $\mathcal{L}[f*g] = F \cdot G$ on next page

Why $\mathcal{L}[f*g] = F(s)G(s)$

$$F(s)\cdot G(s) = \int_0^\infty f(u)e^{-su}\,du \cdot \int_0^\infty g(v)e^{-sv}\,dv$$

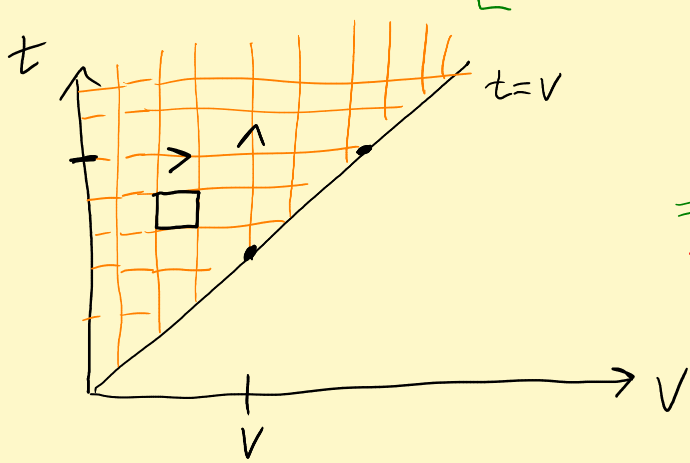
$$\underset{\text{Fubini's}}{=} \int_{v=0}^\infty \left[\int_{u=0}^\infty f(u)g(v)e^{-s(u+v)}\,dv\right] du$$

$$= \int_{v=0}^\infty g(v)\left[\int_{u=0}^\infty f(u)\,e^{-s(\underbrace{u+v}_{t})}\,dv\right] du$$

When $u=0$, $t=v$. $u+v=t$ $u=t-v$

$u\to\infty \Rightarrow t\to\infty$ $dv=dt$

$$= \int_{v=0}^\infty g(v)\left[\int_{t=v}^\infty f(t-v)\,e^{-st}\,dt\right] dv$$



$$\underbrace{\underset{\text{Fubini's}}{=} \int_{t=0}^\infty e^{-st}\left[\underbrace{\int_{v=0}^t g(v)f(t-v)\,dv}_{g*f}\right] dt}_{\mathcal{L}[g*f]} \checkmark$$