Lecture 35 Piecewise defined fcns 7.5 MyLab HW 34

Wed: MyLab HW 35 Fri: MyLab HW 36 & GS HW 34, 35, 36

Formulas from last time

1) $\mathcal{L}[f*g] = F(s)G(s)$ where $(f*g)(t) = \int_0^t f(\tilde{\imath})g(t-\tilde{\imath})\,d\tilde{\imath}$

$$= \int_0^t g(\tilde{\imath})f(t-\tilde{\imath})\,d\tilde{\imath}$$

2) $\mathcal{L}[t f(t)] = -F'(s)$

3) $\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$

4) $\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(\sigma)\,d\sigma$ if f of exponetial type

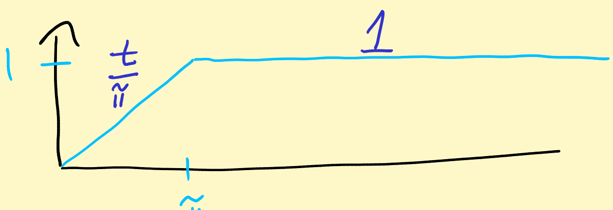
<u>and</u> $\lim_{t\to 0^+} \frac{f(t)}{t}$ exists & is finite.

$$\begin{cases} \text{t-shift rule: } \mathcal{L}[u_c(t)f(t-c)] = e^{-cs}F(s) \\ \text{s-shift rule: } \mathcal{L}[e^{at}f(t)] = F(s-a) \end{cases}$$

<u>Remark</u>: $f(t)$ and $u_0(t)f(t) = u_0(t)f(t-0)$

are the <u>same</u> from point of the Laplace transform

because $\mathcal{L}[f] = \int_0^\infty e^{-st} \underbrace{f(t)}_{u_0(t)f(t) \text{ same on } (0,\infty)}\,dt = F(s)$

<u>Prob</u>: $y'' + y =$ 

$$\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$$

$$= g(t) = \frac{t}{\pi}\left[u_0(t) - u_\pi(t)\right] + 1 \cdot u_\pi(t)$$

$$= u_0(t)\frac{1}{\pi}(t-0) - u_\pi(t)\frac{1}{\pi}\left(\underbrace{(t-\pi)+\pi) + u_\pi(t)}_{\text{cancel}}\right.$$

$$= \underbrace{u_0(t)}_{\text{can remove}} \cdot \frac{t}{\pi} - u_\pi(t)\underbrace{\frac{1}{\pi}(t-\pi)}_{\text{shift of } \frac{1}{\pi}t}$$

So $G(s) = \frac{1}{\pi}\cdot\frac{1}{s^2} - e^{-\pi s}\mathcal{L}\left[\frac{1}{\pi}t\right]$

$$\boxed{G(s) = \frac{1}{\pi s^2} - e^{-\pi s}\cdot\frac{1}{\pi s^2}}$$

$\mathcal{L}$ ODE: $\left(s^2 Y - s\cdot \underset{y(0)}{0} - \underset{y'(0)}{0}\right) + Y = G(s)$

$$Y = \underbrace{\frac{1}{\pi s^2(s^2+1)}}_{H(s)} - e^{-\pi s}\underbrace{\frac{1}{\pi s^2(s^2+1)}}_{H(s)}$$

Step 1 Find $h(t)$

$$H(s) = \frac{1}{\pi s^2(s^2+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{Cs+D}{s^2+1}$$

$$\frac{1}{\pi} = A(s^2+1) + Bs(s^2+1) + (Cs+D)s^2$$

$$\frac{1}{\pi} = \underbrace{(B+C)}_{0}s^3 + \underbrace{(A+D)}_{0}s^2 + \underset{0}{(B)}s + \underset{\frac{1}{\pi}}{A}$$

$C=0$ $D=-\frac{1}{\pi}$ $B=0$

$$H(s) = \frac{1/\pi}{s^2} - \frac{1/\pi}{s^2+1}$$

So $\boxed{h(t) = \frac{1}{\pi}t - \frac{1}{\pi}\sin t}$

$$Y = \underbrace{\frac{1}{\pi s^2(s^2+1)}}_{H(s)} - e^{-\pi s}\underbrace{\frac{1}{\pi s^2(s^2+1)}}_{H(s)}$$

So $\quad y = h(t) - u_\pi(t)\,h(t-\pi)$

$$y = \begin{cases} \frac{t}{\pi} - \frac{1}{\pi}\sin t & 0 \le t \le \pi \\ \left(\frac{t}{\pi} - \frac{1}{\pi}\sin t\right) - \left[\frac{(t-\pi)}{\pi} - \frac{1}{\pi}\underbrace{\sin(t-\pi)}_{-\sin t}\right] & t \geq \pi \end{cases}$$

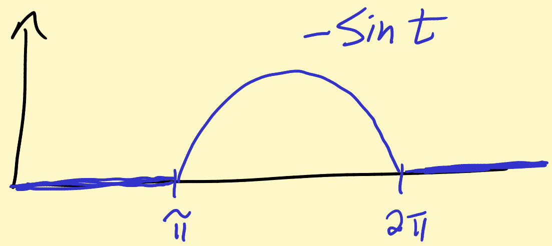
$$= \begin{cases} \frac{t}{\pi} - \frac{1}{\pi}\cdot\sin t & 0 \le t \le \pi \\ 1 - \frac{2}{\pi}\sin t & t \ge \pi \end{cases}$$

Old fashinoned way: Solve IVP on $[0,\pi]$.

Get sol^n y. Solve next IVP at $t=\pi$ using

$\begin{cases} y(\pi) \\ y'(\pi) \end{cases}$ as initial conditions at π. Piece together.

Tricks



$$g(t) = \left[u_\pi(t) - u_{2\pi}(t)\right](-\sin t)$$

$$g(t) = -u_\pi(t)\underbrace{\sin t}_{-\sin(t-\pi)} + u_{2\pi}(t)\underbrace{\sin t}_{\sin(t-2\pi)}$$

$$G(s) = e^{-\pi s}\cdot\frac{1}{s^2+1} + e^{-2\pi s}\cdot\frac{1}{s^2+1}.$$

Or Use $\sin t = \sin(t - \pi + \pi)$

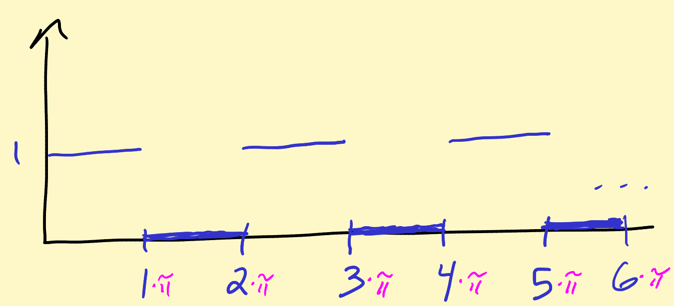
$$= \sin(\underbrace{(t-\pi)}_{\alpha} + \underbrace{\pi}_{\beta}) \quad = -\sin(t-\pi) \checkmark$$

Use $\sin(\alpha+\beta)$ formula. $\sin\pi = 0$

$\cos\pi = -1$

Fun problem

$$y'' + y = g(t) =$$



$$g(t) = \left[u_0(t) - u_\pi(t)\right] + \left[u_{2\pi}(t) - u_{3\pi}(t)\right] + \cdots$$

$$G(s) = \frac{1}{s}\left[1 - e^{-\pi s} + e^{-2\pi s} - e^{-3\pi s} + \cdots\right]$$

$$= \frac{1}{s}\left(1 + \left(-e^{-\pi s}\right) + \left(-e^{-\pi s}\right)^2 + \left(-e^{-\pi s}\right)^3 + \cdots\right)$$

Geom series! $\frac{1}{1-r} = 1 + r + r^2 + \cdots, \ |r| < 1$

$$= \frac{1}{s} \cdot \frac{1}{1-\left(-e^{-\pi s}\right)} = \frac{1}{s\left(1 + e^{-\pi s}\right)}$$

Mistake to clean up here
Makes undoing $\mathcal{L}$ harder

$$Y = \frac{1}{s^2+1} \cdot G(s)$$

$$= \underbrace{\frac{1}{s(s^2+1)}}_{H(s)}\left[1 - e^{-\pi s} + e^{-2\pi s} - e^{-3\pi s} + \cdots\right]$$

$$y = h(t) - e^{-\pi s} h(t-\pi) + e^{-2\pi s} h(t-2\pi) - \cdots$$

Details: $H(s) = \frac{1}{s} - \frac{s}{s^2+1}$

$h(t) = 1 - \cos t$

$$y = \underbrace{\left(u_0 - u_\pi + u_{2\pi} - u_{3\pi} + \cdots\right)}_{\text{square wave}} - \cos t \underbrace{\left(1 + u_\pi(t) + u_{2\pi}(t) + \cdots\right)}_{\text{resonance!}}$$

= something continuous.