Lecture 36 The Dirac Delta fcn 7.6

MyLab HW 35

Fri: myLab HW 36, GS HW 34, 35, 36

Easy fact: $\cos(t-(\text{even})\pi) = \cos t$

$\cos(t-(\text{odd})\pi) = -\cos t$

$h(t) = 1 - \cos t$

$$y(t) = h(t) - e^{-\pi s}h(t-\pi) + e^{-2\pi s}h(t-2\pi) + \cdots$$

$$= \big(\underbrace{u_0 - u_\pi + u_{2\pi} - \cdots}_{\text{square wave}}\big) - \cos t\,\big(\underbrace{1 + u_\pi + u_{2\pi} + \cdots}_{\text{(up) stairs!}}\big)$$

= something continuous! (MAPLE demo)

& continuously diff'ble

Impulse & momentum: $F = ma = m\dfrac{dv}{dt}$

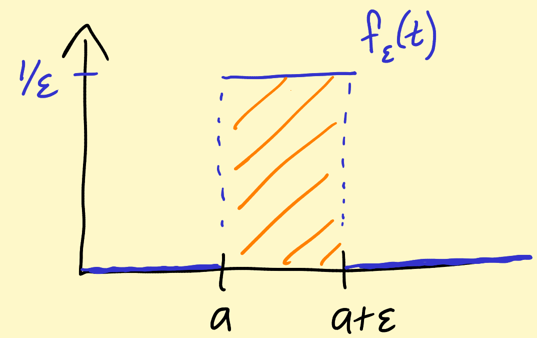
Work/Energy: integrate dx

Imp/moment.: $\int \ldots dt$

$$\int_a^b F(t)\,dt = m\int_a^b \frac{dv}{dt}\,dt = m\,\Delta v$$

Impulse = Δ momentum

Good sledge hammer



Area = 1

(Unit sledge ham.)

Note: $\lim_{\varepsilon\to 0} f_\varepsilon(t) = 0$ for $t \neq a$.

[Ok to think and feel: $f_\infty(a) = \infty$]

$$f_\varepsilon(t) = \frac{1}{\varepsilon}\left[u_a(t) - u_{a+\varepsilon}(t) \right]$$

$$F_\varepsilon(s) = \frac{1}{\varepsilon}\left(\frac{e^{-as}}{s} - \frac{e^{-(a+\varepsilon)s}}{s} \right) = \frac{1}{s}\Bigg[\underbrace{\frac{e^{-as} - e^{-(a+\varepsilon)s}}{\varepsilon}} \Bigg]$$

$$- \frac{h(a+\varepsilon) - h(a)}{\varepsilon}$$

where $h(x) = e^{-xs}$

$\rightarrow -h'(a)$ as $\varepsilon \to 0$

$-(-se^{-as})$

So $\lim_{\varepsilon \to 0} F_\varepsilon(s) = e^{-as}$ (even though f_ε doesn't converge as a fcn!)

<u>Think & feel:</u> $\delta_a(t)$ "$=$" $\lim_{\varepsilon \to 0} f_\varepsilon(t)$

$$\mathcal{L}\left[\delta_a(t) \right] = e^{-as}$$

$\leftarrow$ no rational fcn times this!

<u>Problem</u> $y_\varepsilon'' + y_\varepsilon = f_\varepsilon(t)$ $\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$

$$(s^2+1)\, Y_\varepsilon = \frac{1}{\varepsilon}\left(\frac{e^{-as}}{s} - \frac{e^{-(a+\varepsilon)s}}{s} \right)$$

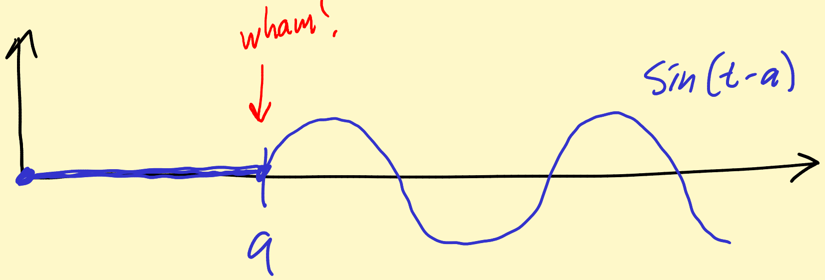
$$Y_\varepsilon = \frac{1}{s^2+1} F_\varepsilon(s) \longrightarrow \frac{1}{s^2+1} e^{-as}$$

as $\varepsilon \to 0$

$$\underline{Y}_\varepsilon \to \underline{Y} = e^{-as}\,\frac{1}{s^2+1}$$

$\uparrow \mathcal{L}[\sin t]$

Ideal sledge hammer: $\quad y = u_a(t)\sin(t-a)$



<u>Note</u> No jump. But the momentum jumps. That's ok.

$\delta_a(t)$ is a "distribution", not a true fcn.

<u>Key properties</u> of $\delta_a(t)$

1) $\delta_a(t) = 0$ if $t \neq a$.

2) $\int \delta_a(t)\,dt = 1$ ← unit impulse

3) If h is continuous, then

$$\int h(t)\delta_a(t)\,dt = h(a)$$

meaning "in the sense of distributions"

$$\lim_{\varepsilon\to 0}\int h(t) f_\varepsilon(t)\,dt = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_a^{a+\varepsilon} h(t)\,dt = h(a)$$

Consequence of (3): $\displaystyle\int_0^\infty e^{-st}\,\delta_a(t)\,dt = e^{-s\cdot a}$

$$\mathcal{L}[\delta_a] = e^{-as}$$

EX $\quad y'' + 4y' + 13y = \delta_\pi(t) \quad \begin{cases} y(0) \\ y'(0) \end{cases}$

$$(s^2+4s+13)\,Y = e^{-\pi s}$$

$$Y = e^{-\pi s}\cdot \frac{1}{s^2+\underline{\underline{4}}s+13}$$

$$= e^{-\pi s}\,\frac{1}{(s+\underset{\frac{1}{2}\cdot 4}{\underset{\shortparallel}{2}})^2+9}$$

$$= e^{-\pi s}\cdot\frac{1}{3}\cdot\frac{3}{(s+2)^2+3^2}$$

$$Y = e^{-\pi s}\,F(s)$$

where $f(t) = \underbrace{\frac{1}{3}e^{-2t}\sin 3t}_{\text{table or s-shift rule}}$

So

$$y(t) = u_\pi(t)\,f(t-\pi) \quad \leftarrow \text{t-shift rule}$$

$$= u_\pi(t)\,\frac{1}{3}e^{-2(t-\pi)}\sin 3(t-\pi)$$

correct

$$= \begin{cases} 0 & t < \pi \\ -\frac{1}{3} e^{2\pi} e^{-2t} \sin 3t & t \geq \pi \end{cases}$$

book answer