Lecture 4 Separable equations (1.4)

Tonight: MyLab HW01, HW02, HW03 at 11:59 pm

Wed night: MyLab HW04

Gradescope written HW02, HW03, HW04 at 11:59 pm

Separable eqn: $\dfrac{dy}{dx} = \dfrac{f(x)}{g(y)}$

E & U Thm Suppose (x_0, y_0) is in a rectangle R and suppose

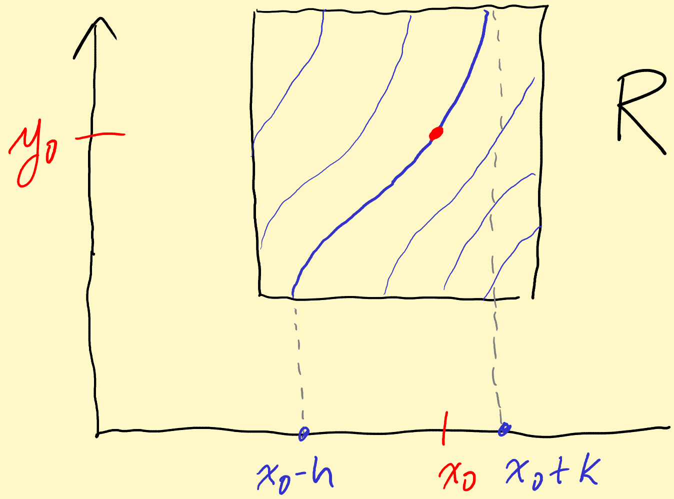
A) $f(x,y)$ is continuous on R and

B) $\dfrac{\partial f}{\partial y}$ is continuous on R.

Then the IVP $\dfrac{dy}{dx} = f(x,y)$ has a unique sol^n in R passing through (x_0, y_0).

E, exists

U

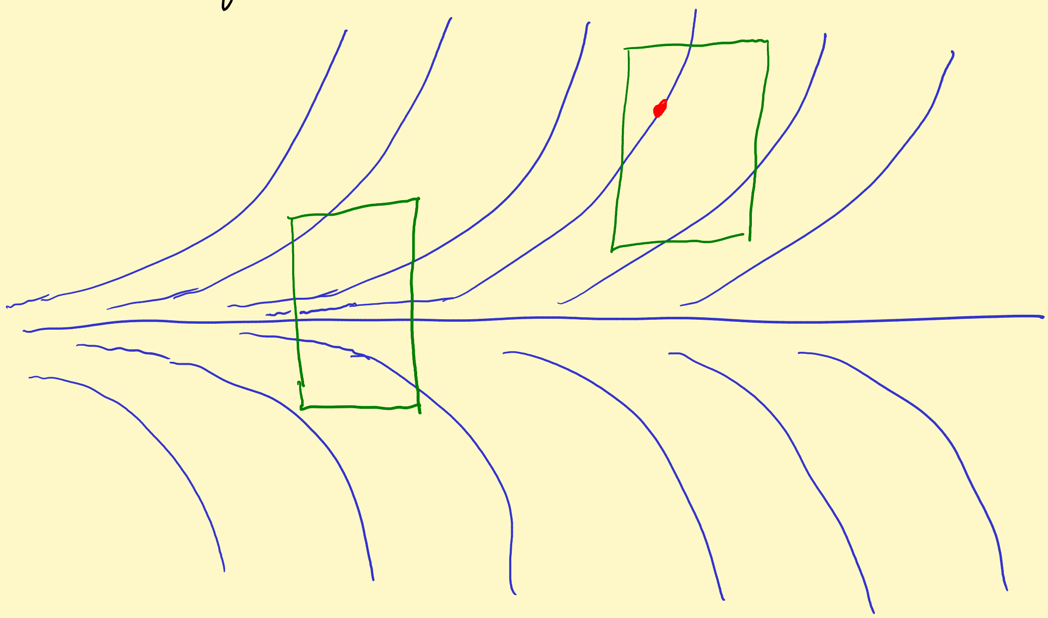


$y(x_0) = y_0$

Note: Sol^n only guaranteed to exist while sol^n stays in R, so on (x_0-h, x_0+k) for some

$h, k > 0$.

EX $\quad \dfrac{dy}{dx} = y^2 \qquad y = \dfrac{-1}{x+c} \leftarrow$ boom! at $x = -c$



Separable eqns $\quad \dfrac{dy}{dx} = \dfrac{f(x)}{g(y)}$

$$\underline{g(y)}\,\frac{dy}{dx} = f(x)$$

$$\frac{d}{dx}\left[G(y)\right] = f(x)$$

Aha! $G(y)$ is an antiderivative of $f(x)$. Write $\int f(x)\,dx = F(x) + C$.

Chain rule

$$\frac{d}{dx}\left[(x^2+1)^5\right] = 5(x^2+1)^4 \cdot (2x)$$

$$\frac{d}{dx}\left(y^5\right) = 5y^4\,\frac{dy}{dx}$$

Hmmm. $\dfrac{d}{dx} G(y) = \underbrace{G'(y)}\,\dfrac{dy}{dx}$

Aha! Want $G'(y) = g(y)$

Need $\boxed{G(y) = \int g(y)\,dy}$

So $G(y) = F(x) + C$.

Now, solve for y to get $y(x)$.

How to remember method

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

Step 1 Separate all x's and y's to one side:

$$g(y)\, dy = f(x)\, dx$$

Step 2 Integrate both sides

$$\int g(y)\, dy = \int f(x)\, dx$$

Step 3

$$\boxed{G(y) = F(x) + C}$$

only one $+C$. ($+c_1$ on left and $+c_2$ on right could be combined into one C.)

Step 4 Solve for $y(x)$.

Important remark To solve IVP $y(x_0) = y_0$, use box: Plug in $x = x_0$ and y_0 in box.

Get $G(y_0) = F(x_0) + C$. ← get C now.

EX $\dfrac{dy}{dx} = \dfrac{y}{x}$

$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

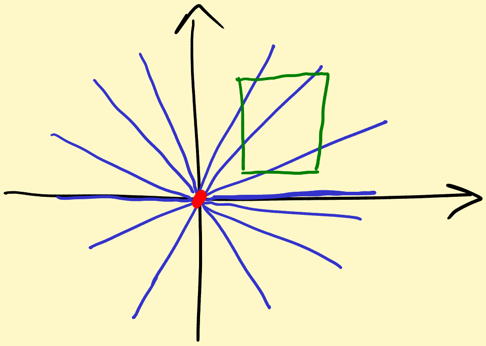
$$e^{\ln|y|} = e^{\ln|x| + C}$$

$$|y| = e^{\ln|x|}e^{C} = |x|e^{C}$$

$$\left|\frac{y}{x}\right| = e^{C}$$

$$\frac{y}{x} = \underbrace{\pm e^{C}}_{\text{a constant, } K} \qquad (K \neq 0)$$

$$\boxed{y = Kx}$$



What the heck is going on at the origin?

$C \to -\infty$, get $K=0$. $y \equiv 0$ missing sol^n

Aha! $\frac{dy}{dx} = \frac{x}{y} \leftarrow f(x,y)$. Nasty when $y=0$

Note: Where $f(x,y)$ is nice, solutions can't cross.

EX $\frac{dy}{dx} = e^{x+y} = e^x e^y = \frac{e^x}{\left(\frac{1}{e^y}\right)} = \frac{e^x}{e^{-y}}$

$$\int \frac{1}{e^y}\, dy = \int e^x \, dx$$

$$\int e^{kx}\, dx = \frac{1}{k} e^{kx} + C$$

$$\int e^{-y}\, dy = e^x + C$$

$$\boxed{\begin{aligned} \ln e^a &= a \\ e^{\ln b} &= b \end{aligned}}$$

$$-e^{-y} = e^x + C$$

$$e^{-y} = -e^x - C$$

$$-y = \ln(-e^x - C)$$

$$y = -\ln(\underline{-C - e^x})$$

better not be negative!

Need $C < 0$.

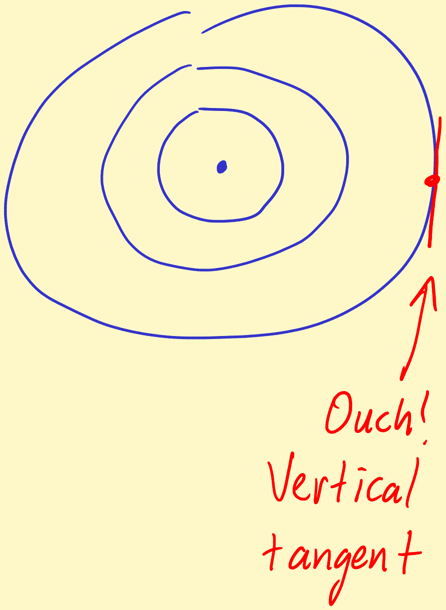
Sol^n blows up when $e^x = -C$.

Problem Sol^{n}s to first order ODE's have <u>one</u> arb. const.

Given a one-parameter family of curves, find an ODE that has them as sol^{n}s.

EX $x^2 + y^2 = C$ ← circles of radius $\sqrt{C}$

$C \geq 0$.



$$x^2 + y^2 = C$$

y = unknown fcn of x

Big idea: Use implicit diff'tion:

$$\frac{d}{dx}\left(x^2 + y^2\right) = \underbrace{\frac{d}{dx}(C)}_{0}$$

$$2x + 2y\frac{dy}{dx} = 0$$

Aha! $\boxed{\dfrac{dy}{dx} = \dfrac{-x}{y}}$

Think in terms of IVP's:

$$\frac{dy}{dx} = \frac{-x}{y} \qquad y(1) = -2$$

$$\int y\,dy = \int -x\,dx$$

$$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$$

$$x^2 + y^2 = 2C \leftarrow \text{circles}$$

Use $y(1) = -2$ to find C <u>now</u>.

$$(1)^2 + (-2)^2 = 2C \qquad \text{Get } C = \frac{5}{2}$$

Sol^n is $\boxed{x^2+y^2=5}$

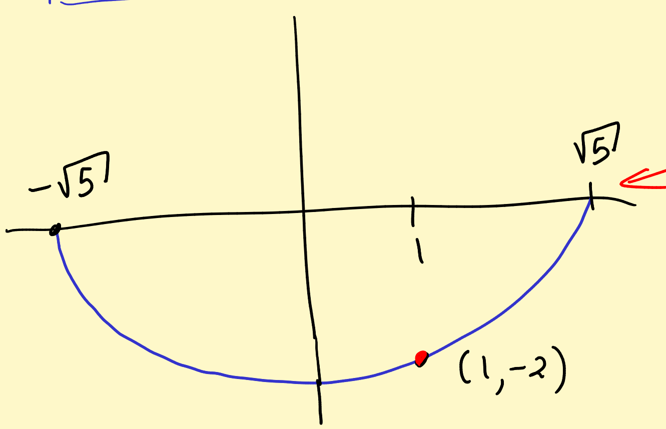
$$y^2 = 5-x^2$$
$$y = \pm\sqrt{5-x^2}$$

What? $\pm$?

Need $y(1) = -2$. $\quad y(1) = \pm\sqrt{5-(1)^2} = \pm 2$

Need the $-$ root.

Ans: $\boxed{y = -\sqrt{5-x^2}}$



Vertical tangent!
Boom! Sol^n is over!

Logistic eq^n $\quad \dfrac{dP}{dt} = kP$

Better model: $\quad \dfrac{dP}{dt} = \left[k(M-P)\right]P$

Separable!