Lecture 5 Linear equations (1.5)

Tonight: MyLab: HW04
Gradescope: HW02, 03, 04
Fri: MyLab HW05
Mon: Labor Day – no class, no HWK

Linear first order ODE: $A(x)\frac{dy}{dx} + B(x)\,y = D(x)$

appear as "linear" terms.

$$\frac{dy}{dx} = \underbrace{-\frac{B(x)}{A(x)}\cdot y + \frac{D(x)}{A(x)}}_{f(x,y)}$$

see: zeroes of $A(x)$ could be bad news!

EX: $\underbrace{x^2\frac{dy}{dx} + 2x\,y}$ $= x^3$

Product rule: $\frac{d}{dx}\left[x^2 y\right]$

So $x^2 y = \int x^3\,dx = \frac{1}{4}x^4 + C$

$$y = \frac{1}{4}x^2 + \frac{C}{x^2}$$

nasty at $x=0$. $A(x)=x^2$ ✓

Method $A(x)\,y' + B(x)\,y = D(x)$

Step 1 Put eqn in Standard Form (divide by $A(x)$)

$$\boxed{\frac{dy}{dx} + P(x)\, y = Q(x)}$$

where $P = \frac{B(x)}{A(x)}$, $Q = \frac{D(x)}{A(x)}$.

Step 2 Compute the "integrating factor" u :

$$u = e^{\int P(x)\,dx}$$

P(x) from Stand. Form

no +C here. Just want one that works.

Step 3 Multiply Standar form eqn by u :

$$\underbrace{u\frac{dy}{dx} + uPy}_{\frac{d}{dx}[uy]} = uQ$$

don't forget u here!

Step 4 $\frac{d}{dx}[u\,y]$ (guaranteed!)

Step 5 Integrate

$$uy = \int u(x)Q(x)\,dx + C$$

+C here.
want all possible sol^{n}'s.

Step 6 $y(x) = \frac{1}{u(x)}\int u(x)Q(x)\,dx + \frac{C}{u(x)}$

Great to have formula. Shows that, on an interval where $P(x)$ and $Q(x)$ are <u>continuous</u>, solutions <u>exist</u> and are continuously differentiable! [Exists on <u>whole</u> interval!]

Also great: Antiderivative of $R(x)$ is $\int_{x_0}^{x} R(t)\,dt$.

Can do numerical integration when P, Q given by data.

But: Use Method and steps instead of formula.

<u>EX</u> $$2\frac{dy}{dx} + y = e^{3x}$$

1) Divide by 2. (<u>S.F.</u>)

$$\frac{dy}{dx} + \underbrace{\frac{1}{2}}_{P(x)} y = \frac{1}{2} e^{3x}$$

2) $$u = e^{\int P(x)dx} = e^{\int \frac{1}{2}dx} = e^{x/2}$$ ← no $+C$

3) $$\underbrace{e^{x/2}\left[\frac{dy}{dx} + \frac{1}{2}y\right]}_{e^{x/2}\frac{dy}{dx} + \left(\frac{1}{2}e^{x/2}\right)y} = e^{x/2}\left(\frac{1}{2}e^{3x}\right) = \frac{1}{2}e^{7x/2}$$

$$\frac{d}{dx}\Big[\underbrace{e^{x/2}}_{u}\,y\Big] \checkmark$$

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$$

4) $$e^{x/2}\,y = \int \tfrac{1}{2}e^{7x/2}\,dx$$
$$= \tfrac{1}{2}\left(\tfrac{2}{7}e^{7x/2}\right) + C$$

5) $$y = \tfrac{1}{7}e^{3x} + Ce^{-x/2} \leftarrow \text{Gen}^{l}\ \text{sol}^{n}$$

IVP: Find sol^n with $y(0)=1$

6) Need $y(0) = \frac{1}{7}e^{3\cdot 0} + Ce^{-0/2}$

Want $1 = \frac{1}{7} + C$

Get $\boxed{C = \frac{6}{7}}$

Sol^n to IVP: $\boxed{y = \frac{1}{7}e^{3x} + \frac{6}{7}e^{-x/2}}$

Why choice of u

$$\frac{dy}{dx} + Py = Q$$

Multiply by a u:

$$\underbrace{u\frac{dy}{dx} + uPy}\ = uQ$$

Looks like

$$\frac{d}{dx}\left[uy\right] = u\frac{dy}{dx} + \frac{du}{dx}y$$

Aha! Need $\boxed{\dfrac{du}{dx} = uP}$ ← separable!

$$\int \frac{1}{u}\,du = \int P(x)\,dx$$

$$\ln|u| = \int P(x)\,dx$$

$$|u| = e^{\int P(x)\,dx}$$

$$u = \pm\, e^{\int P(x)\,dx}$$

← Just want one that works. Take the + one.

EX

$$\frac{dy}{dx} \underbrace{- (\tan x)}_{P(x)}\, y = 8\sin^3 x$$

↑ in standard form.

$$u = e^{\int(-\tan x)\,dx}$$

$$\int \tan x\,dx = \ln|\sec x| + C$$

$$= e^{-\ln|\sec x|}$$

$$e^{-a} = \frac{1}{e^{a}}$$

$$= \frac{1}{e^{\ln|\sec x|}}$$

$$= \frac{1}{|\sec x|} = \frac{1}{\left|\frac{1}{\cos x}\right|} = |\cos x|$$

$$= \pm \cos x$$ ← Pick the + one.

$$\boxed{u = \cos x}$$

$$\underbrace{\cos x\left[\frac{dy}{dx} - (\tan x)\,y\right]}_{\frac{d}{dx}\left[(\cos x)\,y\right]} = \cos x\left(8\sin^3 x\right)$$

$$(\cos x)\,y = 8\int \sin^3 x \underbrace{\cos x\,dx}_{dv}$$

$v = \sin x$

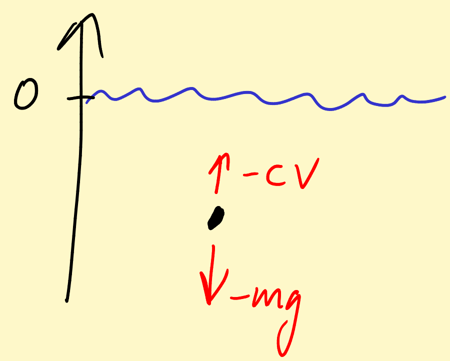
$$= 8\int v^3\,dv = \frac{8}{4}v^4 + C$$

$$(\cos x)\,y = 2\sin^4 x + C$$

$$y = 2\,\frac{\sin^4 x}{\cos x} + \frac{C}{\cos x}$$

$$\boxed{y = 2\tan x\,\sin^3 x + C\sec x}$$

Dropping rock in ocean



$$ma = F$$

$$m\frac{dv}{dt} = -mg - cv$$

$$m\frac{dv}{dt} + cv = -mg \qquad \text{Linear!}$$

Standard form: $$\frac{dv}{dt} + \frac{c}{m}v = -g$$

Int. fact, $$u = e^{\int \frac{c}{m}dt} = e^{\frac{c}{m}t}$$

Mult. by u. LHS: $$\frac{d}{dt}\left[e^{\frac{c}{m}t}\,v\right] = -g e^{\frac{c}{m}t}$$

$\int$:
$$e^{\frac{c}{m}t} v = -\int g e^{\frac{c}{m}t}\, dt$$
$$= -\frac{mg}{c} e^{\frac{c}{m}t} + K$$

So $\boxed{\underset{dy/dt}{v} = -\underbrace{\frac{mg}{c}}_{\text{terminal velocity}} + \underbrace{K e^{-\frac{c}{m}t}}_{\to 0 \text{ as } t \to \infty}}$

Rest at surface: $v(0) = 0$. Get $K = \frac{mg}{c}$

$$\frac{dy}{dt} = -\frac{mg}{c} + \frac{mg}{c} e^{-\frac{c}{m}t}$$

So $y = \int \sim\sim\, dt = \underline{-\frac{mg}{c} t - \frac{m^2 g}{c^2} e^{-\frac{c}{m}t} + K_2}$

$y(0) = 0$. Get $K_2 = \frac{m^2 g}{c^2}$

$$\boxed{y(t) = -\frac{mg}{c} t + \frac{m^2 g}{c^2} - \underbrace{\frac{m^2 g}{c^2} e^{-\frac{c}{m}t}}_{\to 0 \text{ as } t \to \infty}}$$