Lecture 6 Applications

Tonight : MyLab HW05

Mon : Labor Day. No class, no HWK

Wed : MyLab HW06

Gradescope HW05, HW06

Newton's Law of cooling

$T(t)$ = temp of rock at time t

A = temp of ocean

$\frac{dT}{dt}$ proportional to $(T(t)-A)$

$$\frac{dT}{dt} = -k(T-A)$$

$$\frac{dT}{dt} + kT = kA \qquad \text{Linear!}$$

Good advice: If an eqn is linear, that's the method to use!

standard form ✓ $\quad 1 \cdot \frac{dT}{dt} + \underset{\substack{\uparrow \\ P(t)=k}}{\underline{\underline{k}}}\, T = kA$

Int. factor : $\quad u = e^{\int P(t)\,dt} = e^{\int k\,dt} = e^{kt}$ (no $+C$ here)

Use it : $\quad \underbrace{u\cdot\left[T' + kT\right]}_{\frac{d}{dt}\left[\underset{\substack{\uparrow \\ u}}{e^{kt}}\,T\right]} = ukA$

$$\frac{d}{dt}\left[e^{kt}\,T\right] = kAe^{kt}$$

$\int$: $\quad e^{kt}\,T = kA\int e^{kt}\,dt = kA\left[\frac{1}{k}e^{kt}\right] + \underset{\substack{\uparrow \\ +C,\ \text{yes}}}{C}$

Solve for T : $T = A + Ce^{-kt}$

IVP : $T(0) = T_0$

Plug in $t=0$: $\underbrace{T(0)}_{\text{want } = T_0} = A + Ce^{-k\cdot 0} = A + C$

See $T_0 = A + C$

so $\boxed{C = T_0 - A}$

Sol^n : $T(t) = A + (T_0 - A)e^{-kt}$

Prob : 40° outside. Find stiff at 65° at time t_F. When was time of death?

Sol^n : $A = 40$. Let $t=0$ be time of death.

Also know $T(0) = 98.6$

$$T(t) = A + (T_0 - A)e^{-kt}$$
$$= 40 + (58.6)e^{-kt}$$

Know $T(t_F) = 40 + 58.6\, e^{-kt_F} \overset{\text{want}}{=} 65$

$$58.6\, e^{-kt_F} = 25$$

$$e^{-kt_F} = \frac{25}{58.6}$$

Take Ln of both sides $\quad -kt_F = \ln \frac{25}{58.6}$

Hmmm. Get kt_F. Want t_F.

What to do?

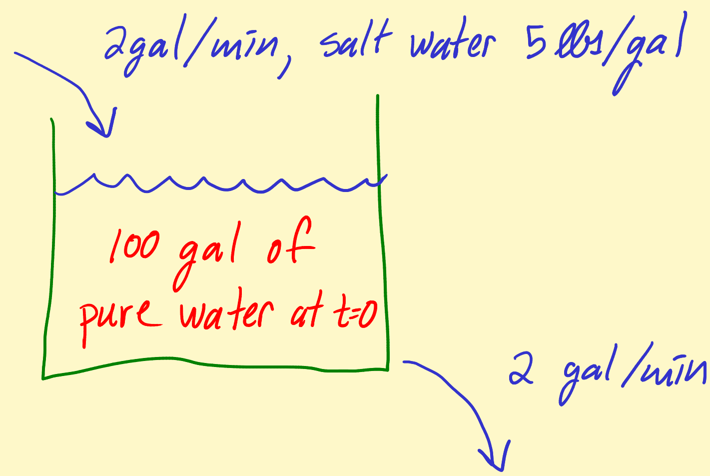
Aha! Idea! Wait an hour and measure temp again!

Get formula for $-k(t_F+1)$.

$$\begin{cases} -kt_F = \alpha \leftarrow \ln \frac{25}{58.6} \\ -k(t_F+1) = \beta \leftarrow \text{know too} \end{cases}$$

Two equns, two unknowns. Solve for both t_F and k.

Mixing poblems



How much salt is in tank at time t.

Let $S(t)$ = amount of salt in tank at time t in lbs.

Note: $S(0) = 0$ (pure)

Hmmm. Rates: $\frac{dS}{dt} = (\text{Rate in}) - (\text{Rate out})$

$$= (2\ \text{gal/min})(5\ \text{lbs/gal}) - (2\ \text{gal/min}) \cdot \left[\frac{S(t)}{100\ \text{gals}}\right]$$

Linear!
$$\frac{dS}{dt} = 10 - \frac{1}{50} S$$

$$\boxed{\frac{dS}{dt} + \frac{1}{50} S = 10}$$

S.F. ✓

IVP: $S(0) = 0$

$$u = e^{\int \frac{1}{50} dt} = e^{t/50}$$

$$\frac{d}{dt}\left[e^{t/50} S\right] = 10 e^{t/50}$$

$$e^{t/50} S = 500 e^{t/50} + C$$

$$S(t) = 500 + Ce^{-t/50}$$

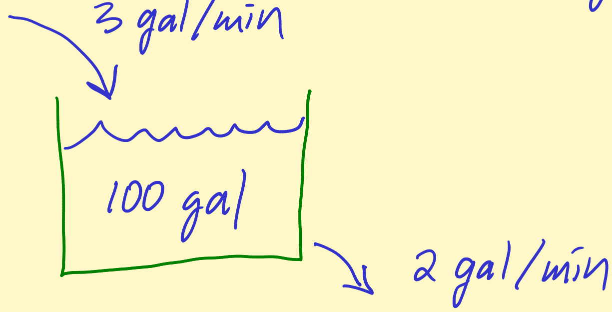
$S(0) = 0$ requires $C = -500$

Ans: $S(t) = 500 - 500 e^{-t/50}$

Note: Concentration at time t: $c(t) = \dfrac{S(t)}{100 \text{ gal}}$

$$\text{Rate salt goes out} = (2 \text{gal/min}) \frac{S(t)}{100 \text{ gal}}$$

What if vol of tank were changing?



Same work, except $\cdots \cdot \left[\dfrac{S(t)}{V(t)} \right]$

changing volume

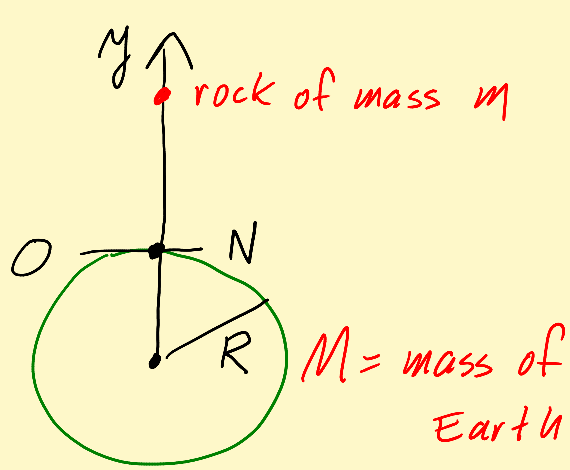
Easy: $V(t) = 100 + 1 \cdot t$

or: $V(0) = 100$

$$\frac{dV}{dt} = (\text{Rate in}) - (\text{Rate out})$$

$$= 3 \text{ gal/min} - 2 \text{ gal/min} = 1, \text{ etc.}$$

Shoot a rock into space



$$F = ma$$

$$-G \frac{Mm}{(y+R)^2} = m \frac{dv}{dt}$$

Ouch! Too many variables! y, t, v

Clever trick (method). Use the chain rule.

$$\boxed{\frac{dv}{dt} = \frac{dv}{dy}\cdot\underbrace{\frac{dy}{dt}}_{v}} = v\,\frac{dv}{dy}$$

Aha! Eqn is:

$$\boxed{-\frac{GMm}{(y+R)^2} = mv\,\frac{dv}{dy}}$$

Separable!

$$\int -\frac{GMm}{(y+R)^2}\,dy = \int mv\,dv$$

$$\frac{GMm}{y+R} = \frac{1}{2}mv^2 + C$$

grav. potential eng.   kinetic energy

Use $v(0) = v_0$ to get C.

Get $C = \frac{GM}{R} - \frac{1}{2}v_0^2$.

Solve for $y+R$:

$$\boxed{y+R = \frac{GM}{\frac{1}{2}v^2 + \left(\frac{GM}{R} - v_0\right)^2}}$$

Solve for v. Get separable eqn: $\underbrace{\frac{dy}{dt}}_{v} = \text{fcn of } y$

Ugh!

<u>Ques</u> How high does rock go?

Aha! $v=0$ at peak

Get y_{max} from red box.

Fascinating: $y_{max} \to \infty$ as $v_0 \to \sqrt{\frac{GM}{R}}$

escape velocity!