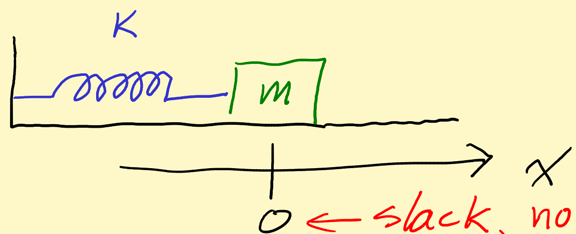
# Lecture 7 Homogeneous eqns, Bernoulli eqns, Reduction of order

Due tonight : MyLab: HW 06
Gradescope: Written HW 05, HW 06

New office hours: T, Th 2:00-3:00 pm in MATH 750 and via Piazza

Reduction of order Turn a $2^{nd}$ order problem into a $1^{st}$ order problem we know how to do.

EX : Work-energy trick




← slack, no friction

$F = ma$ ← second order

$$\underbrace{-kx}_{\text{Hooke's law}} = m\frac{d^2x}{dt^2} = m\frac{dv}{dt} \underset{\text{chain rule}}{=} m\frac{dv}{dx}\cdot\underbrace{\frac{dx}{dt}}_{v} = mv\frac{dv}{dx}$$

Separable!

$$\int -kx\,dx = \int mv\,dv$$

$$\underbrace{-\tfrac{1}{2}kx^2}_{\text{Potential energy of spring}} = \underbrace{\tfrac{1}{2}mv^2}_{\text{Kinetic energy}} + C$$

$$\Delta(\text{Pot. energy}) = \Delta(\text{Kinetic energy})$$

To continue : Solve for $v$

$$v^2 = \frac{2}{m}\left(-C - \tfrac{1}{2}kx^2\right)$$

$$v = \sqrt{A \pm Bx^2}$$

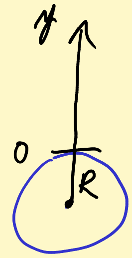
$\uparrow$

$\frac{dx}{dt}$ Separable

$$\int \frac{dx}{\sqrt{A \pm Bx^2}} = \int dt = t + C$$

$\uparrow$ Arc sines and cosines!

Rock into space




Got:

$$\frac{GMm}{(y+R)} = \frac{1}{2}v^2 + \left(\frac{GM}{R} - \frac{1}{2}v_0^2\right)$$

$$y_{max} \to \infty \text{ as } v_0 \to \sqrt{\frac{2GM}{R}}$$

$\left( v=0 \text{ at } y_{max} \right)$

$\uparrow$ Escape velocity

To continue: Assume $v_0 = v_e = \sqrt{\frac{2GM}{R}}$. $y=0$ when $t=0$.

$$\frac{GM}{(y+R)} = \frac{1}{2}v^2$$

Solve for $v$: $$v = \frac{\sqrt{2GM}}{(y+R)^{1/2}}$$

$\uparrow$

$\frac{dy}{dt}$ Separable!

$$\int (y+R)^{1/2}\, dy = \int \sqrt{2GM}\, dt$$

$$\frac{1}{\left(\frac{3}{2}\right)} (y+R)^{3/2} = \sqrt{2GM}\, t + C$$

$y=0$ when $t=0$ : $\frac{2}{3} R^{3/2} = 0 + C$

$$y+R = \left(\frac{3}{2}\sqrt{2GM}\, t + R^{3/2}\right)^{2/3}$$

---

**Method of reduction of order** $\qquad \dfrac{d^2y}{dx^2} = F\left(x, y, \dfrac{dy}{dx}\right)$

A) $\qquad \dfrac{d^2y}{dx^2} = F\left(x, \dfrac{dy}{dx}\right) \longleftarrow$ y missing

Trick: Let $p = \dfrac{dy}{dx}$. Then $\dfrac{d^2y}{dx^2} = \dfrac{dp}{dx}$

$$\underbrace{\frac{d^2y}{dx^2}} = F\left(x, \frac{dy}{dx}\right)$$

$$\frac{dp}{dx} = F(x, p) \longleftarrow \text{1}^{\text{st}} \text{ order ODE}$$

EX:
$$\frac{d^2y}{dx^2} + \underbrace{\frac{dy}{dx}}_{p} = e^x \quad \leftarrow y \text{ missing}$$

$$\frac{dp}{dx} + p = e^x \quad \leftarrow \text{Linear } 1^{st} \text{ order !}$$

$$u = e^{\int 1\,dx} = e^x$$

$$\underbrace{e^x\left(\frac{dp}{dx} + p\right)}_{[e^x p]'} = e^x \cdot e^x = e^{2x}$$

$$e^x p = \int e^{2x}\,dx = \frac{1}{2}e^{2x} + C$$

$$\underbrace{p}_{\frac{dy}{dx}} = \frac{1}{2}e^x + Ce^{-x}$$

$$y = \int \frac{1}{2}e^x + Ce^{-x}\,dx$$

$$= \frac{1}{2}e^x - Ce^{-x} + K$$

B) $\dfrac{d^2y}{dx^2} = F\left(y, \dfrac{dy}{dx}\right) \leftarrow x$ missing

$$p = \frac{dy}{dx}$$

$$\underbrace{\frac{dp}{dx}}_{} = F(y, p) \leftarrow$$ too many vars. Use chain rule trick!

$$\frac{dp}{dy} \cdot \underbrace{\frac{dy}{dx}}_{p}$$

$$\boxed{p\frac{dp}{dy} = F(y, p)} \leftarrow 1^{st} \text{ order}$$

Solve it. Get $p = f(y)$

$$\frac{dy}{dx} = f(y) \leftarrow \text{Separable!}$$

## Change of variables

EX $\quad \frac{dy}{dx} = (x + y + 8)^2$

$y$ = unknown fcn of $x$

Big idea: Let $v = x + y + 8$

also unknown fcn of $x$

$$\frac{dy}{dx} = v^2 \leftarrow \text{too many vars.}$$

Hmmm: $\quad v = x + y + 8$

$\frac{dy}{dx}$ on LHS. Solve for $\quad y = v - x - 8$

So $\quad \frac{dy}{dx} = \frac{dv}{dx} - 1 - 0$

$$\frac{dy}{dx} = v^2$$

$$\frac{dv}{dx} - 1 = v^2$$

Separable!

$$\frac{dv}{dx} = 1 + v^2$$

$$\int \frac{dv}{1+v^2} = \int dx$$

$$\tan^{-1} v = x + c$$

$$v = \tan(x+c)$$

Finally $\quad x + y + 8 = \tan(x+c)$

$$\boxed{y = -8 - x + \tan(x+c)}$$

## Homogeneous eqns $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$

Let $\boxed{v = \frac{y}{x}}$

Solve for $y$: $y = xv$

$$\frac{dy}{dx} = \frac{d}{dx}\left[xv\right] = 1 \cdot v + x\frac{dv}{dx}$$

$$\boxed{\frac{dy}{dx} = v + x\frac{dv}{dx}}$$

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

$$v + x\frac{dv}{dx} = F(v)$$ ← Separable!

EX: $$\frac{dy}{dx} = e^{y/x} + \frac{y}{x}$$

$$v + x\frac{dv}{dx} = e^{v} + v$$

$$x\frac{dv}{dx} = e^{v}$$

$$\int e^{-v}\,dv = \int \frac{dx}{x}$$

$$-e^{-v} = \ln|x| + C$$

Ln:

$$e^{-v} = -\ln|x| - C$$

$$-v = \ln\left(-\ln|x| - c\right)$$

$\frac{y}{x}$

$$y = -x\ln\left(-\ln|x| - c\right)$$

EX $\quad \dfrac{x-y}{x+y} \cdot \dfrac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \dfrac{1-\frac{y}{x}}{1+\frac{y}{x}}$ homog!

EX $\quad \ln y - \ln x = \ln \frac{y}{x}$ homog!

<u>Bernoulli eqn</u> $\quad \dfrac{dy}{dx} + P(x)\,y = Q(x)\,y^n$

Change of var: $\quad v = y^{1-n}$

Use chain rule, plug into eqn, get new eqn:

$$\boxed{\frac{dv}{dx} + (1-n)P(x)\,v = (1-n)Q(x)}$$

Linear

EX: Put in standard form ← Step 1

$$1 \cdot \frac{dy}{dx} - \frac{3}{2x} y = 2x\, y^{-1} \leftarrow \boxed{n=-1}$$

$$P(x) = -\frac{3}{2x} \qquad Q(x) = 2x$$

$$v = y^{1-n} = y^{1-(-1)} = y^2$$

New eqn $\quad \frac{dv}{dx} + (1-(-1))\underbrace{\left(-\frac{3}{2x}\right)}_{P(x)} v = (1-(-1))\underbrace{2x}_{Q(x)}$

$$\frac{dv}{dx} - \frac{3}{x} v = 4x$$

Solve for $v$.

$v = y^2$. So $y = \sqrt{v}$