Lecture 8 Exact eqns (1.6)

Due tonight: MyLab HW 07

Bernoulli eqns: $\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$

Change of var: $v = y^{1-n}$ ⟵ $y = v^{1/(1-n)}$

[Bernoulli: Try $y = v^p$]

New linear eqn: $\boxed{\frac{dv}{dx} + (1-n)P(x)\,v = (1-n)\,Q(x)}$

EX $\frac{dy}{dx} - \frac{3}{2x}\,y = 2x\,y^{-1}$ ⟵ $n=-1$

$v = y^{1-(-1)} = y^2$ $\quad \boxed{v = y^2}$

New eqn: $\frac{dv}{dx} + (1-\underset{n=-1}{(-1)})\underbrace{\left[-\frac{3}{2x}\right]}_{P(x)} v = (1-\underset{n}{(-1)})\underbrace{\left[2x\right]}_{Q(x)}$

$\boxed{\frac{dv}{dx} - \frac{3}{x}\,v = 4x}$ Linear!

$$u = e^{\int -\frac{3}{x}\,dx} = e^{-3\ln x} = e^{\ln x^{-3}} = \frac{1}{x^3}$$

$$\underbrace{u\,[LHS]} = u \cdot 4x$$

$$\frac{d}{dx}\left[\underset{u}{\frac{1}{x^3}} \cdot v\right] = \frac{1}{x^3} \cdot 4x = \frac{4}{x^2}$$

$$\frac{1}{x^3} \cdot v = \int \frac{4}{x^2}\,dx = -\frac{4}{x} + C$$

$$V = -4x^2 + Cx^3$$
$$\parallel$$
$$y^2$$

$$y = \pm\sqrt{-4x^2 + Cx^3}$$

Choose C, $\pm$ via I.V.P.

Exact equations

Solve ODE. Then solve for C:

EX $\quad x^2y + y^3 = C \leftarrow$ family of curves

Defines $y(x)$ implicitly.

Find ODE: $$\frac{d}{dx}\left[x^2y + y^3\right] = \frac{d}{dx}C = 0$$

$$\left(2x\cdot y + x^2\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0$$

$$(2xy) + (x^2 + 3y^2)\frac{dy}{dx} = 0$$

Want to go other way.

Chain rule $$\frac{d}{dx}\varphi(u,v) = \frac{\partial\varphi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\varphi}{\partial v}\frac{\partial v}{\partial x}$$

Suppose $\varphi(x, y) = C$

$u = x \quad v = y(x)$

$$\frac{\partial \varphi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial v}\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial \varphi}{\partial x}\cdot\underbrace{\frac{\partial x}{\partial x}}_{1} + \frac{\partial \varphi}{\partial y}\cdot\frac{\partial y}{dx} = 0$$

ODE : $\boxed{\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx} = 0}$

Hmmm: Given ODE: $\boxed{M(x,y) + N(x,y)\frac{dy}{dx} = 0}$

Aha! Need φ such that $\begin{cases} \frac{\partial \varphi}{\partial x} = M \\ \frac{\partial \varphi}{\partial y} = N \end{cases}$

Then $\varphi(x,y) = C$ solves ODE!

Famous problem Given a force field $\vec{F}$, when is $\vec{F}$ conservative, meaning Work = $W = \int_\gamma \vec{F}\cdot d\vec{s}$ is path independent?

Answer A) When $\vec{F} = \nabla\varphi$ ← φ potential fcn

$\Longleftrightarrow$ B) Curl $\vec{F} \equiv 0$ ← 3-D version

2-D version: $\vec{F} = M\hat{\imath} + N\hat{\jmath}$

Want $\vec{F} = \nabla\varphi$

Want $\dfrac{\partial\varphi}{\partial x} = M \qquad \dfrac{\partial\varphi}{\partial y} = N$

Hmmm $\underbrace{\dfrac{\partial}{\partial y}\left(\dfrac{\partial\varphi}{\partial x}\right)}_{\frac{\partial^2\varphi}{\partial y\,\partial x}} = M_y \qquad \underbrace{\dfrac{\partial}{\partial x}\left(\dfrac{\partial\varphi}{\partial y}\right)}_{\frac{\partial^2\varphi}{\partial x\,\partial y}} = N_x$

should be = (φ C^2 smooth)

Necessary condition: $\boxed{\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}}$ ← exactness criterion

2-D curl: $\vec{F} = M\hat{\imath} + N\hat{\jmath} + 0\cdot\hat{k}$

$$\operatorname{curl}\vec{F} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$$

MATH 261: On a rectangle, the red box is also <u>sufficient</u>.

How to remember: $M + N\frac{dy}{dx} = 0$

Multiply by dx: $\underbrace{M\,dx + N\,dy}_{\text{differential form}} = 0$

$$d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy \leftarrow \text{"total differential"}$$

Fact If $M\,dx + N\,dy = d\varphi$, it is called an exact differential

Exactness criterion $M + N\frac{dy}{dx} = 0$

write $M\,dx + N\,dy = 0$

$$\boxed{\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}}$$

$$\frac{\partial}{\partial x}\left[\begin{matrix}\text{Thing by}\\ dy\end{matrix}\right] = \frac{\partial}{\partial y}\left[\begin{matrix}\text{Thing by}\\ dx\end{matrix}\right]$$

EX $\underbrace{(y^2 + \cos x)}_{M} + \underbrace{(2xy + \sin y)}_{N}\frac{dy}{dx} = 0$

Exact?

$$\begin{cases} \dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left(2xy + \sin y\right) = 2y + 0 \\[2ex] \dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left(y^2 + \cos x\right) = 2y + 0 \end{cases} \quad = \checkmark$$

Go ahead and look for φ with

$$\begin{cases} \dfrac{\partial \varphi}{\partial x} = M = y^2 + \cos x & (A) \\[2ex] \dfrac{\partial \varphi}{\partial y} = N = 2xy + \sin y & (B) \end{cases}$$

Step 1 Use up (A):

$$\varphi = \int \underbrace{y^2 + \cos x}_{M} \, dx$$

"partial integral with respect to x". Treat y as a const.

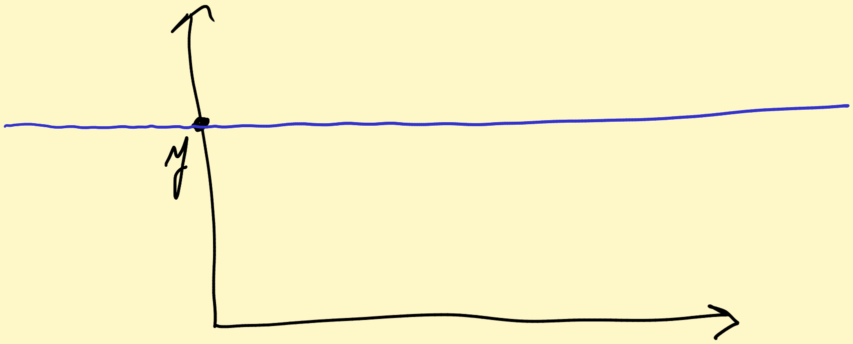
$$= x y^2 + \sin x \quad + \underbrace{C(y)}$$

unknown function of other variable y

Why $C(y)$ instead of just C?

If φ_1 and φ_2 both satisfied $\frac{\partial \varphi_j}{\partial x} = M$, then

$$\frac{\partial}{\partial x}\left(\varphi_1 - \varphi_2\right) \equiv 0$$



$\varphi_1 - \varphi_2 \equiv$ const on horiz lines. Const can depend on y.

Also check that $\frac{\partial}{\partial x}\left(\varphi_1 + C(y)\right) = \frac{\partial \varphi_1}{\partial x}$ ✓

(A) is used up: $\boxed{\varphi = xy^2 + \sin x + C(y)}$

Step 2 Use (B) in a different way

Want $\quad \frac{\partial \varphi}{\partial y} = N$

$$\frac{\partial}{\partial y}\Big[\underbrace{xy^2 + \sin x + C(y)}_{\varphi \text{ from Step 1}}\Big] \overset{\text{want}}{=} \underbrace{2xy + \sin y}_{N}$$

Need: $\quad 2xy + 0 + C'(y) = 2xy + \sin y$

Miracle! due to exactness. All x's cancel

Need $\quad \boxed{C'(y) = \sin y}$

Step 4 $\quad C(y) = \int \sin y \, dy = -\cos y + K$

can skip K here

Step 5 $\quad \varphi(x,y) = xy^2 + \sin x + \left[-\cos y + K\right]$

Step 6 $\quad$ Ans: $\quad \varphi(x,y) = C$

$$xy^2 + \sin x - \cos y + K = C$$

K

Step 7 $\quad$ Solve for y if you can.

If not, this equation defines $y(x)$ implicitly.