

Lecture 9 Population dynamics (2.1)

MyLab: HW 08 tonight

$$\frac{dP}{dt} = kP \quad \text{Malthusian}$$

Logistic eqn: $\frac{dP}{dt} = [k(M-P)] \cdot P$

Separable! $\int \frac{dP}{(M-P)P} = \int k dt$

$$= kt + C$$

Partial fractions! $\frac{1}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$

Step 1 Mult by denom:

$$1 = A(M-P) + BP$$

$$[0]P + 1 = [-A + B]P + [AM]$$

$$AM = 1$$

$$A = 1/M$$

$$B - A = 0$$

$$B = A = \frac{1}{M}$$

$$\int \frac{dP}{(M-P)P} = \int k dt$$

$$\int \frac{(1/M)}{P} + \frac{(1/M)}{\underbrace{M-P}_{u=M-P, du=-dP}} dP = kt + C$$

$$\frac{1}{M} \ln P - \frac{1}{M} \ln |M-P| = kt + C$$

$$\ln \frac{P}{|M-P|} = kMt + MC$$

Exponentiate both sides:

$$\frac{P}{|M-P|} = e^{kMt} e^{MC}$$

$|M-P| = \pm(M-P)$

$$\frac{P}{M-P} = \underbrace{(\pm e^{MC})}_C e^{kMt}$$

$$P = (M-P) C e^{kMt}$$

$$P(1 + C e^{kMt}) = M C e^{kMt}$$

$$P = \frac{Mc e^{kMt}}{1 + C e^{kMt}} \cdot \frac{e^{-kMt}}{e^{-kMt}}$$

$$P = \frac{CM}{C + e^{-kMt}}$$

Can see that $\lim_{t \rightarrow \infty} P(t) = \frac{CM}{C} = M$

Remark : It is also a Bernoulli eqn!

$$\frac{dP}{dt} = k(M-P)P = kMP - kP^2$$

$$\frac{dP}{dt} - \underbrace{kMP}_{p = -kM} = \underbrace{-kP^2}_{q = -k} \quad \boxed{n=2}$$

$$V = P^{1-n} = P^{1-2} = P^{-1} = \frac{1}{P} \quad \boxed{V = \frac{1}{P}}$$

New eqn: $\frac{dV}{dt} + \underbrace{(1-n)p(t)}_{(1-2)(-kM)} V = \underbrace{(1-n)q(t)}_{(1-2)(-k)}$

$$\frac{dv}{dt} + kmv = k \quad \text{easy!}$$

Get v . Then $p = \frac{1}{v}$. Same answer!

$$\text{IVP: } P(0) = P_0$$

$$P_0 = \frac{cM}{c + e^{-km \cdot 0}}$$

$$P_0 = \frac{cM}{c + 1}$$

$$P_0(c+1) = cM$$

$$P_0 = (M - P_0)c$$

$$\boxed{c = \frac{P_0}{M - P_0}}$$

$$P(t) = \frac{P_0 M}{P_0 + (M - P_0)e^{-kMt}}$$

Autonomous first order ODEs

$$\frac{dP}{dt} = f(P) \leftarrow \text{no time } t \text{ on RHS}$$

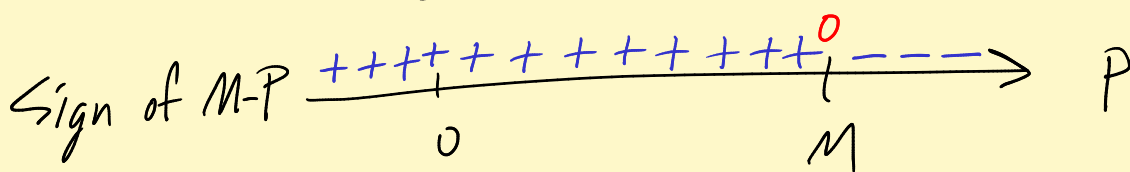
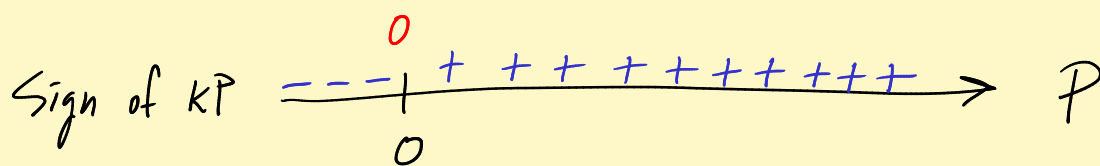
Direction fields are easy! (Isoclines horiz lines.)

$$\frac{dP}{dt} = k(M-P)P \leftarrow f(P)$$

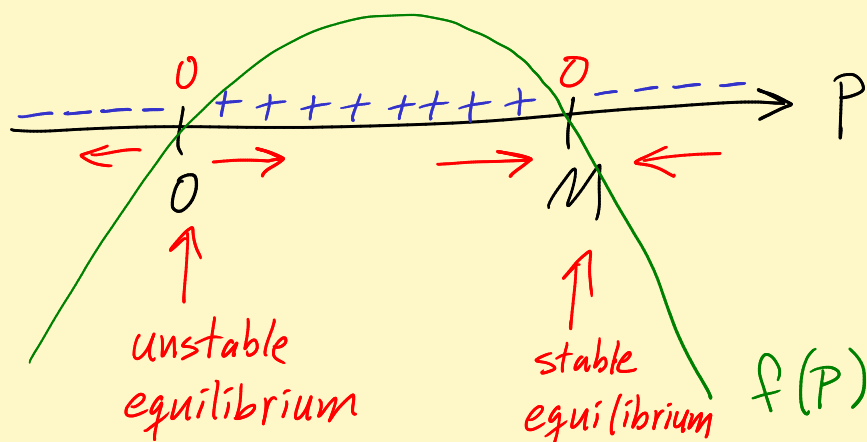
Sign of $\frac{dP}{dt}$: $\frac{dP}{dt} > 0$ P growing

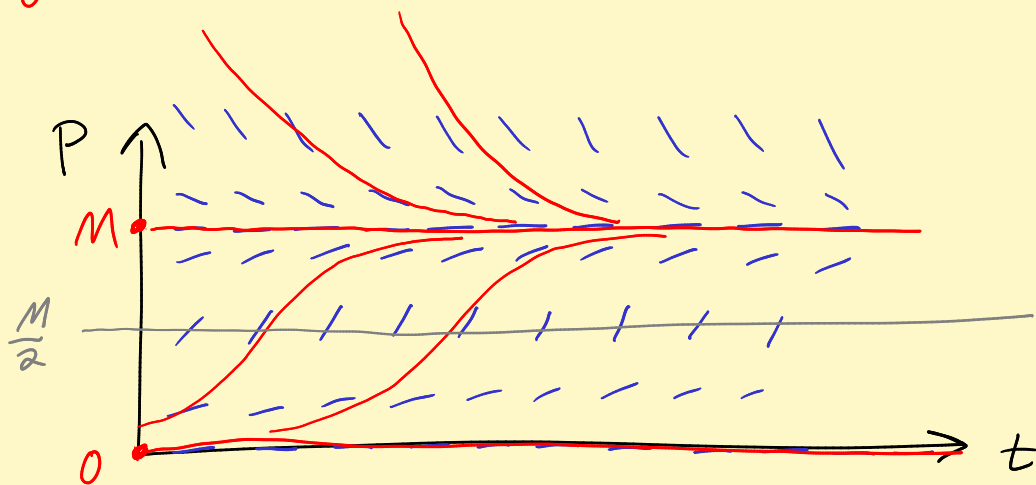
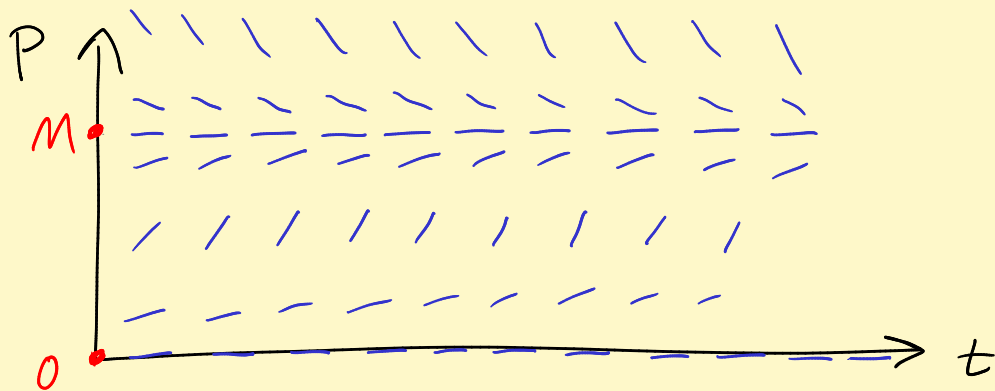
$\frac{dP}{dt} < 0$ P shrinking

Graph Sign of $\frac{dP}{dt}$ v.s. P



Sign of $\frac{dP}{dt} = k(M-P)P$





Fun thing : Where are inflection pts in $0 < P < M$?

Hmmm :

$$\frac{dP}{dt} = k(M-P)P = kMP - kP^2$$

$$\text{So } \frac{d^2P}{dt^2} = kM \frac{dP}{dt} - 2kP \frac{dP}{dt}$$

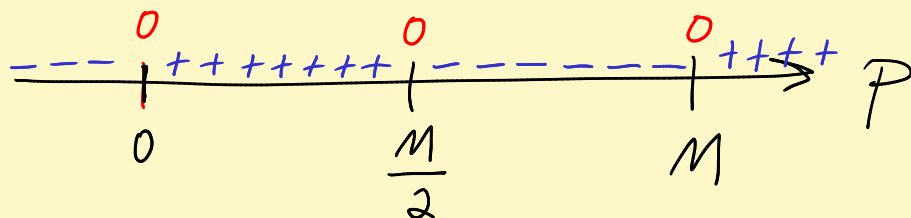
$$\text{Aha! } \frac{dP}{dt} = k(M-P)P$$

$$= kM \cdot k(M-P)P - 2kP \cdot k(M-P)P$$

$$= k(M-P)P [kM - 2kP]$$

$$= k^2(M-2P)(M-P)P$$

Sign of $\frac{d^3P}{dt^2}$



↑ inflection pt
happens when solⁿs
pass through $P = \frac{M}{2}$ horiz. line