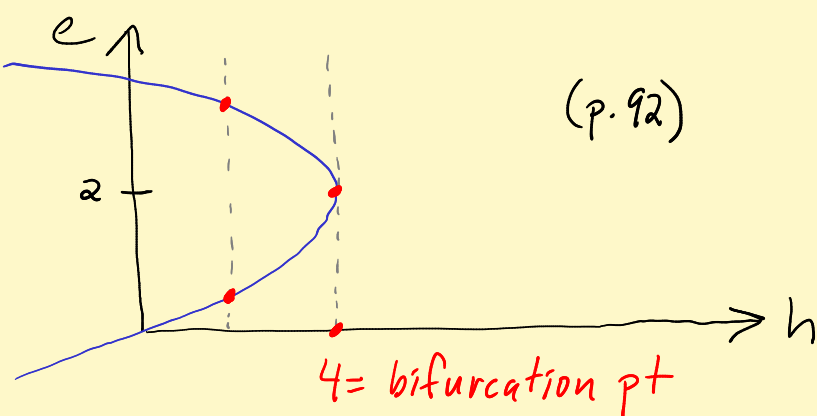


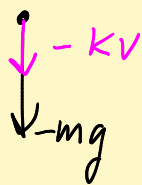
Bifurcation diagram: $\frac{dy}{dx} = y(4-y) - h$



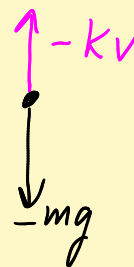
equilibrium sol^{ns} $y \equiv e$ when
 e is a root of $e(4-e) - h = 0$
 $e = 2 \pm \sqrt{4-h}$

EX Laminar flow: Stoke's Law: Friction force = $-k \cdot v$

Up $v \uparrow$



Down $v \downarrow$



$$F = ma$$

$$-mg - kv = m \frac{dv}{dt} \leftarrow \text{Linear!}$$

Did this. Got terminal velocity. Happens when
 friction force = weight

$$|-kv| = mg$$

$$|v_t| = \frac{mg}{k}$$

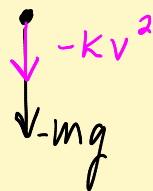
Going down:

$$v_t = -\frac{mg}{k}$$

EX Higher velocity ; Friction force $\propto v^2$

2

Up $v \uparrow$



Down $v \downarrow$



Two different problems!

Up $F = ma$

$$-mg - kv^2 = m \frac{dv}{dt} \quad \leftarrow \text{separable}$$

$$\int \frac{m dv}{mg + kv^2} = \int -dt = -t + C$$

Table $\int \frac{1}{1+u^2} du = \tan^{-1} u + C$

$$\frac{m}{mg} \int \frac{dv}{1 + \left(\frac{k}{mg}\right) v^2} = -t + C$$

Aha! Change of vars : $u^2 = \left(\frac{k}{mg}\right) v^2$

$$\boxed{u = \sqrt{\frac{k}{mg}} v}$$
$$du = \sqrt{\frac{k}{mg}} dv$$

Get $\boxed{\sqrt{\frac{m}{gk}} \tan^{-1} \left(\sqrt{\frac{k}{mg}} v \right) = -t + C}$

Get C at this pt! Initial cond. $V = V_0 > 0$
When $t=0$ and $y=0$.

Want v as a fcn of t .

$$\tan^{-1} \left(\sqrt{\frac{k}{mg}} v \right) = \sqrt{\frac{gk}{m}} (-t + C)$$

$$\sqrt{\frac{k}{mg}} v = \tan \left(\sqrt{\frac{gk}{m}} (-t + C) \right)$$

$\uparrow$
 $v = \frac{dy}{dt}$

$$y = \int \sqrt{\frac{mg}{k}} \tan \left(\sqrt{\frac{gk}{m}} (-t + C) \right) dt$$

From table

$$\int \tan u \, du = \ln |\sec u| + C$$

Do it!

Hmmm, $v=0$ at max height. Can easily get time at $y_{\max}$. Do 2nd integral. Plug $t_{\max}$ to get $y_{\max}$.

Down

$$F = ma$$

$$-mg + kv^2 = m \frac{dv}{dt}$$

$$\int \frac{dv}{g - \frac{k}{m}v^2} = \int -dt = -t + C$$

Use $\int \frac{du}{1-u^2} = \tanh^{-1}u + C$

↑ preferable

or $\frac{1}{1-u^2} = \frac{1}{(1-u)(1+u)} = \frac{A}{1-u} + \frac{B}{1+u}$

What is the terminal velocity?

When friction force = weight

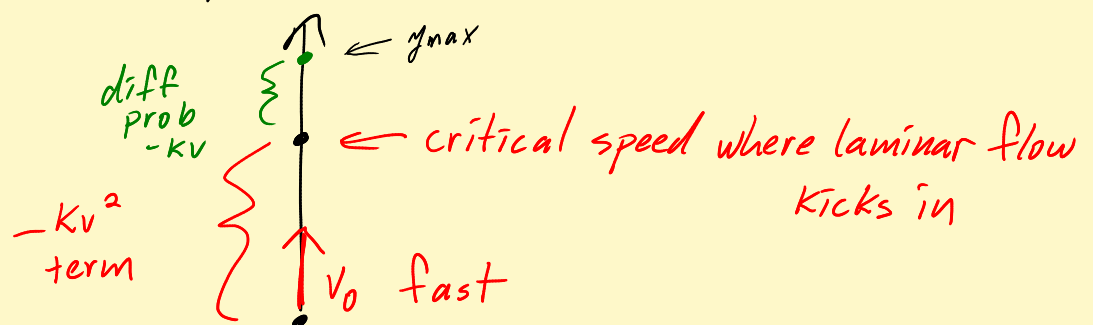
$$kv^2 = mg$$

$$|v| = \sqrt{\frac{mg}{k}}$$

Down: $v = -\sqrt{\frac{mg}{k}}$

Real life Boundary between laminar flow and

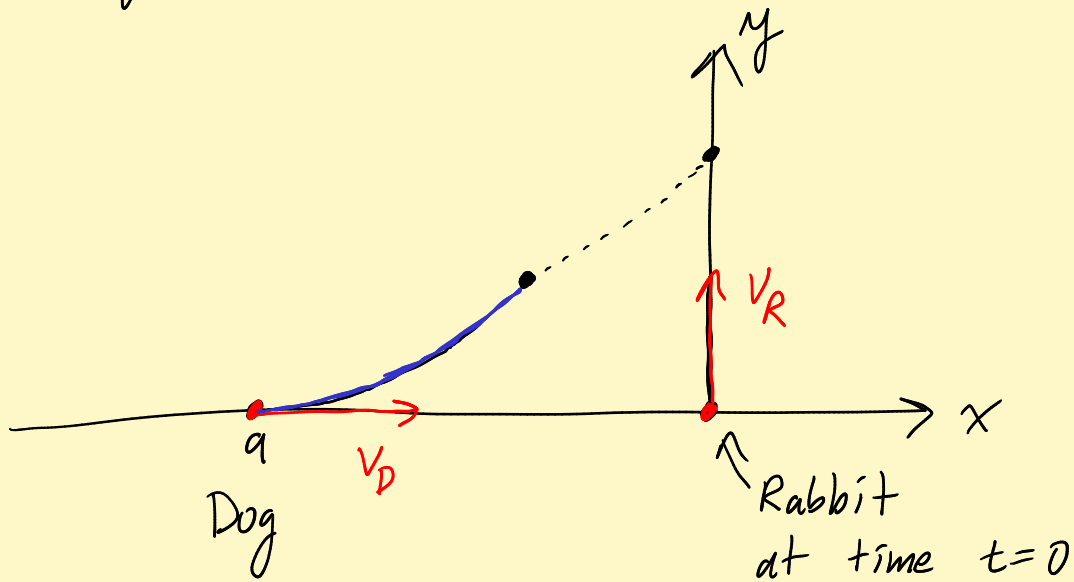
turbulence:



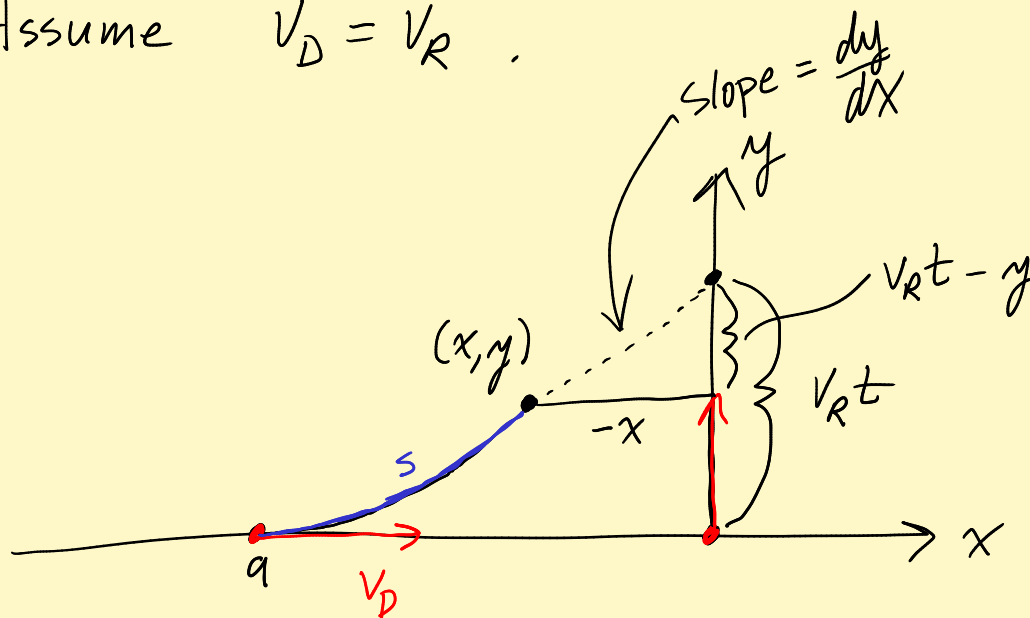
Dog chasing rabbit problem

Pursuit path

5



Assume $v_D = v_R$.



Know $s = \int_a^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

$$\begin{cases} \frac{ds}{dt} = v_D \\ s = v_D t \\ t = \frac{s}{v_D} \end{cases}$$

Know $\frac{dy}{dx} = \frac{v_R t - y}{-x}$

$$= \frac{\frac{v_R}{v_D} \int_a^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx - y}{-x}$$

Hmmm. Solve for $\int_a^x$. Diff!

$$y - x \frac{dy}{dx} = \frac{v_R}{v_D} \int_a^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Diff:

$$\frac{dy}{dx} - \frac{dy}{dx} - x \frac{d^2 y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

no y !

$$-x \frac{d^2 y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Let $P = \frac{dy}{dx}$.

$$-x \frac{dP}{dx} = \sqrt{1 + P^2}$$

separable!

$$\int \frac{dP}{\sqrt{1 + P^2}} = - \int \frac{dx}{x}$$

$$\sinh^{-1} P = -\ln|x| + C$$

Find C . $\frac{dy}{dx}(0) = 0$, $y(0) = 0$ when $x = a$.

$$\underbrace{\sinh^{-1} 0}_0 = -\ln|a| + C$$

$$C = \ln|a|$$

$$p = \sinh(\ln|a| - \ln|x|)$$

$$\frac{dy}{dx} = \sinh\left(\ln\left|\frac{a}{x}\right|\right)$$

$\frac{a}{x}$ both < 0

$$\sinh u = \frac{1}{2}(e^u - e^{-u})$$

Aha! $\frac{dy}{dx} = \frac{1}{2}\left(\frac{a}{x} - \frac{x}{a}\right)$ ← Wow!

$$y = \frac{a}{2} \ln|x| - \frac{1}{4a} x^2 + K$$

Get K from $y=0$ when $x=a$.