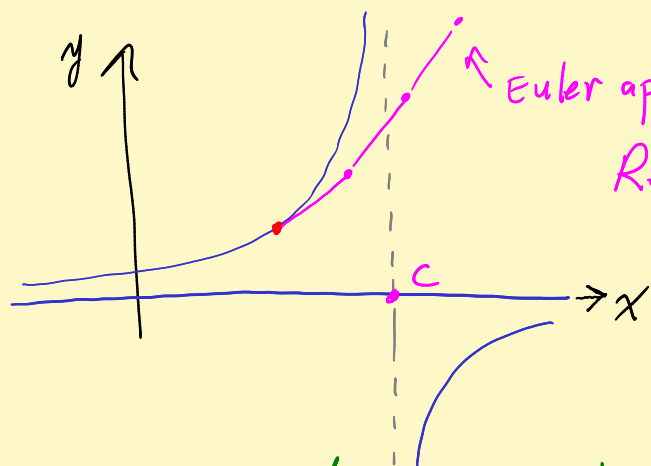


Lecture 13 Second order ODE (3.1)

MyLab HW 12

GS written HW 10, 12

Euler method fail: $\frac{dy}{dx} = y^2 \leftarrow$ Recall solⁿs $y = \frac{1}{c-x}$ blow up (no 11)



Euler approx. Clueless about blow-up!
Reduce h . $y(c)$ just gets bigger and bigger.

$F=ma$ is a 2nd order ODE!

Easiest 2nd order ODE: $\frac{d^2 y}{dx^2} = x^3 \leftarrow$ (linear)

$$\frac{dy}{dx} = \frac{1}{4} x^4 + C_1$$

$$y = \underbrace{\frac{1}{20} x^5}_{\substack{\uparrow \\ \text{a particular} \\ \text{sol}^n \text{ to } y''=x^3}} + \underbrace{C_1 x + C_2}_{\substack{\text{general sol}^n \text{ to} \\ \text{the "homog eqn"} \\ y''=0}}$$

2nd order linear ODE:

$$A(x)y'' + B(x)y' + C(x)y = D(x)$$

Standard form: Divide out by $A(x)$

$$(+)\quad y'' + P(x)y' + Q(x)y = R(x)$$

where $P(x) = \frac{B(x)}{A(x)}$ and ...

↑ zeroes of $A(x)$ can be a problem!

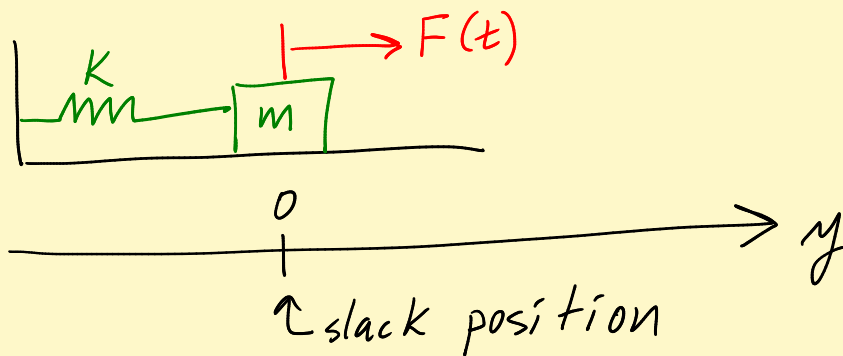
E. & U. Thm (+) plus two initial conditions

$$\begin{cases} y(x_0) = y_0 \\ y'(x_0) = y_0' \end{cases}$$

has a unique solⁿ
 ↑
 solⁿ exists
 (E.)
 (U.)

on any interval containing x_0 where $P(x), Q(x), R(x)$ are all continuous.

EX $y'' + 3y' + 2y = 0 \leftarrow \text{homogeneous eqn}$
 (*)



$$F = ma$$

$$F(t) - \underbrace{K y}_{\text{Hooke's}} - c \frac{dy}{dt} = m \frac{d^2 y}{dt^2}$$

$$m y'' + c y' + K y = F(t)$$

Could try reduction of order:

$$y'' + 3y' + 2y = 0 \leftarrow \text{no } t!$$

$$p = \frac{dy}{dt} : \quad \frac{dp}{dt} + 3p + 2y = 0$$

Chain rule

$$p \frac{dp}{dy}$$

Get separable eqn.

Solve. Get $p = f(y)$.

$$\frac{dp}{dt} = \frac{dp}{dy} \cdot \underbrace{\frac{dy}{dt}}_p$$

$$\frac{dy}{dt} = f(y)$$

Also separable. Solve it.

Ugh!

Euler's idea

Guess

$$y = e^{rt}$$

$$y' = re^{rt}$$

$$y'' = r^2 e^{rt}$$

$$\text{So } y'' + 3y' + 2y = (r^2 e^{rt}) + 3(re^{rt}) + 2(e^{rt}) = 0$$

$$= \underbrace{[r^2 + 3r + 2]}_{\text{needs to be 0}} e^{rt} = 0$$

want
↓
↑ never zero

Characteristic eqn: $r^2 + 3r + 2 = 0$

$$(r+2)(r+1) = 0$$

Need $r = -2, -1$

Get two sol^{ns} $y_1 = e^{-2t}$, $y_2 = e^{-t}$

Superposition principle for linear homogeneous

2nd order eqns: If y_1 and y_2 solve (*),

then so do linear combos $c_1 y_1 + c_2 y_2$.

why: $L[y] = y'' + P y' + Q y$

$$L[\underbrace{c_1 y_1 + c_2 y_2}_{\text{sol}^n!}] = (c_1 y_1'' + c_2 y_2'') + P(c_1 y_1' + c_2 y_2') + Q(c_1 y_1 + c_2 y_2)$$

$$= c_1 \underbrace{L[y_1]}_{=0} + c_2 \underbrace{L[y_2]}_{=0} = 0$$

Aha! $y = c_1 e^{-2t} + c_2 e^{-t}$ solves (*).

Big question: Is this the gen^l solⁿ to (*)?

Equivalent question (because of E. & U. Thm):

Can we solve any and all initial value problems

(IVP) $\begin{cases} y(t_0) = A \\ y'(t_0) = B \end{cases}$ for any choice of A, B ?

Hmmm. $\begin{cases} y = c_1 e^{-2t} + c_2 e^{-t} \\ y' = -2c_1 e^{-2t} - c_2 e^{-t} \end{cases}$ Take $t_0 = 0$.

IVP $\begin{cases} y(0) = c_1 + c_2 = A \\ y'(0) = -2c_1 - c_2 = B \end{cases}$ want

$$\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

Cramer's rule

$$c_1 = \frac{\det \begin{bmatrix} A & 1 \\ B & -1 \end{bmatrix}}{\det \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}} \leftarrow \text{hope } \neq 0$$

$$c_2 = \frac{\det \begin{bmatrix} 1 & A \\ -2 & B \end{bmatrix}}{\det \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}}$$

$$\det \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} = 1(-1) - 1(-2) = 1$$

So $c_1 = -A - B$ $c_2 = B + 2A$

Yes! $c_1 e^{-2t} + c_2 e^{-t}$ is the gen^l solⁿ to (*).

General problem

$$\begin{cases} y = c_1 y_1 + c_2 y_2 \\ y' = c_1 y_1' + c_2 y_2' \end{cases}$$

IVP

$$\begin{cases} y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0) = A \\ y'(t_0) = c_1 y_1'(t_0) + c_2 y_2'(t_0) = B \end{cases}$$

$$\underbrace{\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

Need the "Wronskian" determinant

$$W = W[y_1, y_2] = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

$$= y_1 y_2' - y_2 y_1' \quad \text{to be non-zero at } t=t_0.$$

EX :

$$\det \begin{bmatrix} e^{-2t} & e^{-t} \\ -2e^{-2t} & -e^{-t} \end{bmatrix} = -e^{-3t} - (-2e^{-3t}) = e^{-3t}$$

↑ never zero!

y_1 and y_2 above give gen^l solⁿ to any and all IVP's with any initial t_0 . 7

Bummers: Roots of Char eqn : r_1, r_2

- 1) Repeated roots. Only get y_1 .
- 2) $r = a \pm bi$ complex roots. ?