

Lecture 14 2nd order linear ODE with constant coefficients

1

Tonight: MyLab HW 13

Existence & Uniqueness Thm (E. & U.) Suppose $P(x), Q(x), R(x)$

are continuous on an interval (a, b) and $x_0 \in (a, b)$. Then the IVP

$$y'' + P(x)y' + Q(x)y = R(x) \quad (+)$$

$\uparrow$ non-homogeneous, $R \neq 0$.

with $\begin{cases} y(x_0) = y_0 \\ y'(x_0) = y_0' \end{cases}$ has a unique solⁿ that is

twice continuously differentiable on (a, b) .

Homogeneous problem: $y'' + P(x)y' + Q(x)y = 0 \quad (*)$

Last time: If we find two solutions y_1 and y_2 to $(*)$

with $W(x_0) = \det \begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \neq 0,$

then $c_1 y_1 + c_2 y_2$ is the gen^l solⁿ to $(*)$.

Cool fact: Abel's formula shows that, if $y_1 \neq$

y_2 solve $(*)$, then $W[y_1, y_2] \equiv 0$ or never zero!

Why:

$$\begin{aligned} & y_1 \cdot [y_2'' + P y_2' + Q y_2 = 0] \\ - & y_2 \cdot [y_1'' + P y_1' + Q y_1 = 0] \end{aligned}$$

$$\underbrace{(y_1 y_2'' - y_2 y_1'')}_{\frac{dW}{dx}} + P \underbrace{(y_1 y_2' - y_2 y_1')}_{W} = 0$$

$$\uparrow \text{check: } \frac{d}{dx}(y_1 y_2' - y_2 y_1') = \checkmark$$

$$\frac{dW}{dx} + PW = 0 \leftarrow \text{Linear!}$$

$$u = e^{\int P dx}$$

$$\underbrace{u(W' + PW)}_{[uW]'} = u \cdot 0 = 0$$

$$uW = C$$

$$e^{\int P dx} \cdot W = C$$

$$W = C e^{-\int P(x) dx}$$

$\uparrow$ never zero

$\leftarrow$ Abel's formula

Two cases: 1) $C = 0$. Then $W \equiv 0$

2) $C \neq 0$. Then W never zero.

Important consequence: Only need to test the Wronskian

at one pt x . Pick an easy one (like $x_0=0$)!

Meaning of $W(x_0)=0$

Think about system

$$\begin{cases} c_1 y_1(x_0) + c_2 y_2(x_0) = A = y_0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) = B = y_0' \end{cases}$$

$$\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

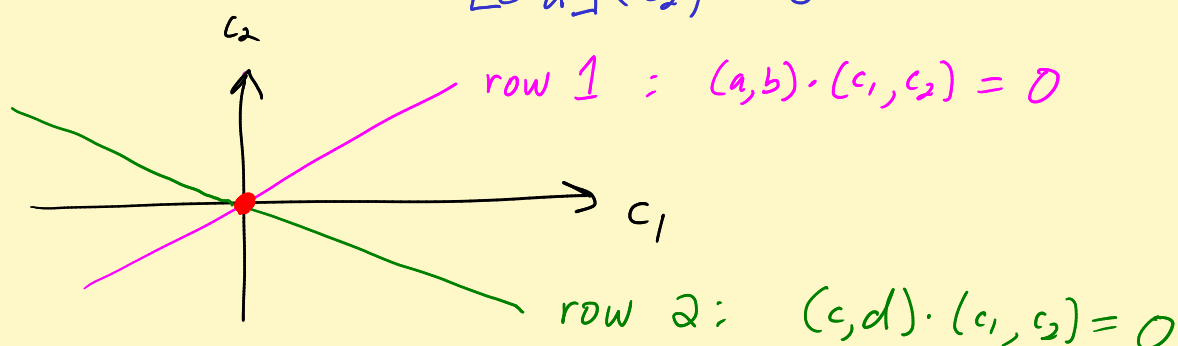
If $\det \neq 0$, get unique solⁿs
 c_1, c_2 for any given choice of A, B .

(via Cramer's rule).

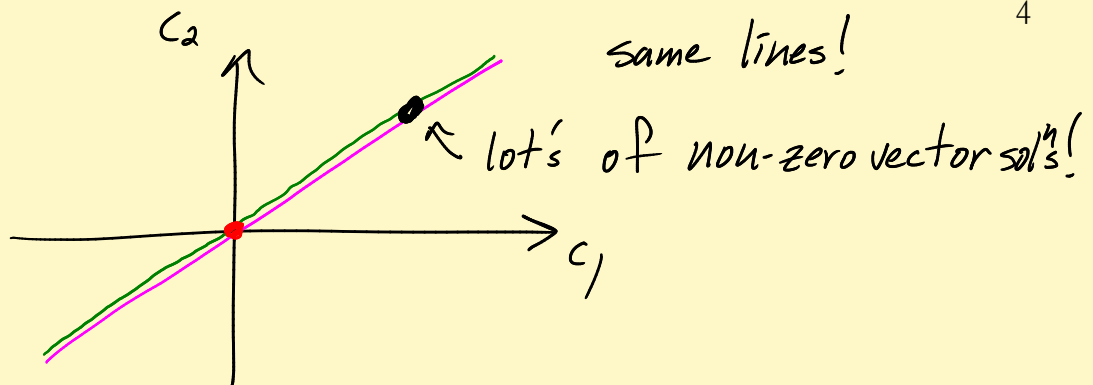
Determinant and system $A \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \vec{0}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \vec{0}$$

$\det A \neq 0$
picture



$\det A = 0$
picture



Math 265 facts : $A\vec{x} = \vec{0}$, A square matrix

- 1) Solution space $\{\vec{x} : A\vec{x} = \vec{0}\}$
is a vector space.
- 2) $A\vec{x} = \vec{0}$ has the unique solⁿ $\vec{x} = \vec{0}$
if and only if $\det A \neq 0$.
- 3) $A\vec{x} = \vec{0}$ has non-zero vector solⁿs
if and only if $\det A = 0$.

So, if $W(x_0) = 0$, there are constants k_1 and k_2
not both zero $\left[\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \neq \vec{0} \right]$ such that

$$\begin{cases} k_1 y_1(x_0) + k_2 y_2(x_0) = 0 & (A) \\ k_1 y_1'(x_0) + k_2 y_2'(x_0) = 0 & (B) \end{cases}$$

Hmmm. (A),(B) means $y = k_1 y_1 + k_2 y_2$ satisfies

$$\begin{cases} y(x_0) = 0 \\ y'(x_0) = 0 \end{cases}$$

Aha! $y \equiv 0$ is "another" solⁿ that satisfies the same IVP. So U of $E, \dot{\in} U$. Then

implies that $0 \equiv k_1 y_1 + k_2 y_2$

Suppose $k_2 \neq 0$. Then $y_2 = -\frac{k_1}{k_2} y_1$.

Fancy word: y_1 and y_2 are linearly dependent.

Fact $W(x_0) = 0 \iff y_1$ and y_2 are lin. dep.

Consequence Only need to find two solⁿs y_1 and y_2 to (*) that are "different" in the sense

that $\frac{y_2}{y_1}$ is not a constant.

EX: e^{-x} and e^{-2x} are "different". $\frac{e^{-x}}{e^{-2x}} = e^x \leftarrow \text{not const.}$

e^{3x} and $7e^{3x}$ are not "different" enough.⁶

Last time

$$ay'' + by' + cy = 0 \quad (*) \quad a, b, c \text{ constants}$$

Tried $y = e^{rx}$.

$$\underbrace{(ar^2 + br + c)}_{\text{needs to be 0}} e^{rx} = 0$$

$\uparrow$ never zero $\uparrow$ want

Characteristic eqn: $\boxed{ar^2 + br + c = 0}$

Best case: Two distinct real roots

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad b^2 - 4ac > 0$$

$$\frac{e^{r_1 x}}{e^{r_2 x}} = e^{\overbrace{(r_1 - r_2)}^{\neq 0} x} \leftarrow \text{not a const.}$$

Fancy word: $e^{r_1 x}$ and $e^{r_2 x}$ are linearly independent
(not lin. dep.).

$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ is gen^l solⁿ to (*).

Bummer case #1: Repeated roots

$$r_1 = \frac{-b \pm 0}{2a} = -\frac{b}{2a} \quad b^2 - 4ac = 0$$

Only get one solⁿ. $y_1 = e^{r_1 x}$ where $r_1 = -\frac{b}{2a}$

Ouch! Need a second lin. indep. solⁿ y_2 .

Euler

$$Ly = ay'' + by' + cy$$

$$L[e^{rx}] = \underbrace{(ar^2 + br + c)}_{a(r-r_1)^2} e^{rx}$$

Take $\frac{\partial}{\partial r}$

$$\frac{\partial}{\partial r} L[e^{rx}] = \frac{\partial}{\partial r} [a(r-r_1)^2 e^{rx}]$$

$$\begin{aligned} &= L\left[\frac{\partial}{\partial r} e^{rx}\right] = \underbrace{2a(r-r_1)e^{rx} + a(r-r_1)^2 [xe^{rx}]}_{\text{Aha! Let } r=r_1} \\ &= L[xe^{rx}] \end{aligned}$$

$$\text{Euler : } L[xe^{rx}] = 0$$

$\uparrow$
 $y_2 = xe^{r_1 x}$ also solves (*)!

$$\frac{y_2}{y_1} = \frac{xe^{r_1 x}}{e^{r_1 x}} = x \leftarrow \text{not a constant!}$$

Fact: Repeated roots $r_1, r_2 = \frac{-b}{2a}$

Gen^l solⁿ is

$$y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$$

Bigger bummer case

$$r_1, r_2 = \alpha \pm \beta i$$

$$b^2 - 4ac < 0$$

$$= \frac{-b}{2a} \pm w i$$

$$w = \frac{\sqrt{|b^2 - 4ac|}}{2a}$$

Euler: Pretended not to notice! Invented the complex exponential function. e^z , z complex.