

Lecture 15 2nd order linear ODE with const. coeff (part 2)

MyLab 14 tonight

Exam 1, Thurs, Oct 8, 8:00-9:00 pm [Prob. 1-7 multiple choice, 8-10 written]

$$\text{Euler: } e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots + \frac{z^n}{n!} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\begin{aligned} e^{x+iy} &= e^x \cdot e^{iy} \\ &= e^x \cdot \left[1 + (iy) + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \dots \right] \\ &= e^x \left[\underbrace{\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots \right)}_{\cos y} + i \underbrace{\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots \right)}_{\sin y} \right] \end{aligned}$$

$$\boxed{e^{x+iy} = (e^x \cos y) + i (e^x \sin y)} \quad \leftarrow \text{Defines } e^z, z \in \mathbb{C}.$$

Last time

$$ay'' + by' + cy = 0 \quad \uparrow \text{homog}$$

$$\text{Try } y = e^{rx} : \quad \underbrace{(ar^2 + br + c)}_{\text{need } = 0} e^{rx} = 0$$

$$\text{Get } r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Case 1 r_1, r_2 real and $\neq 0$. $b^2 - 4ac > 0$

$$\text{Genl Soln: } y = \underline{c_1 e^{r_1 x} + c_2 e^{r_2 x}}$$

Case 2 repeated (real) roots

$$b^2 - 4ac = 0$$

2

$$r_1, r_2 = -\frac{b}{2a}$$

$$\text{Gen}^l \text{ sol}^n : \quad \underline{y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}}$$

Case 3 Complex roots : $b^2 - 4ac < 0$

$$r_1, r_2 = \alpha \pm \beta i$$

$$\alpha = -\frac{b}{2a}$$

$$\beta = \frac{\sqrt{|b^2 - 4ac|}}{2a}$$

$$\text{Gen}^l \text{ sol}^n : \quad \underline{y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x}$$

What Euler did

$$\text{Complex sol}^n : \quad \underline{Y(x) = u(x) + i v(x)}$$

$$\text{Define } Y'(x) = \lim_{h \rightarrow 0} \frac{Y(x+h) - Y(x)}{h}$$

$$= \lim_{h \rightarrow 0} \left[(DQ u) + i (DQ v) \right]$$

$$= u'(x) + i v'(x)$$

Do it again : $\overline{Y}''(x) = u''(x) + i v''(x)$.

Key facts A complex solⁿ $\overline{Y} = u + i v$ to (*)

yields that u and v are two real solⁿs to (*).

Why : $L[y] = a y'' + b y' + c y$ is linear and

a, b, c are real numbers.

$$\text{So } \underbrace{L[\overline{Y}]}_{\text{red bracket}} = a \overline{Y}'' + b \overline{Y}' + c \overline{Y}$$

$\overline{Y} \text{ sol}^n \rightarrow 0$

$$= a(u'' + i v'') + b(u' + i v') + c(u + i v)$$

$$= \underbrace{L[u]}_0 + i \underbrace{L[v]}_0$$

u, v real solⁿs!

Key fact #2

Think: $r = \alpha + \beta i$

$$\frac{d}{dx} \left[e^{(\alpha + \beta i)x} \right] = (\alpha + \beta i) e^{(\alpha + \beta i)x}$$

Write out defⁿs of both sides and see they are equal!

and

$$\frac{d^2}{dx^2} \left[e^{(\alpha + \beta i)x} \right] = (\alpha + \beta i)^2 e^{(\alpha + \beta i)x}$$

$$\text{So } L[e^{rx}] = (ar^2 + br + c)e^{rx} \text{ even } r \in \mathbb{C}!$$

Aha! If $r = \alpha + \beta i$ is a complex root, get

$$L \left[\underset{\substack{\uparrow \\ \text{complex sol}^n}}{e^{(\alpha + \beta i)x}} \right] = 0$$

$$\begin{aligned} Y &= e^{(\alpha + \beta i)x} = e^{\alpha x + \beta x i} = e^{\alpha x} e^{i \beta x} \\ &= e^{\alpha x} (\cos \beta x + i \sin \beta x) \\ &= \underbrace{(e^{\alpha x} \cos \beta x)}_U + i \underbrace{(e^{\alpha x} \sin \beta x)}_V \quad \text{real sol}^n \end{aligned}$$

Get $y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$

by super position principle.

Is it the gen^l solⁿ?

Wronskian?

$$W[y_1, y_2] = \det \begin{bmatrix} e^{\alpha x} \cos \beta x & e^{\alpha x} \sin \beta x \\ ()' & ()' \end{bmatrix}$$

$$\det \begin{bmatrix} e^{\alpha x} \cos \beta x & e^{\alpha x} \sin \beta x \\ \alpha e^{\alpha x} \cos \beta x - \beta e^{\alpha x} \sin \beta x & \alpha e^{\alpha x} \sin \beta x + \beta e^{\alpha x} \cos \beta x \end{bmatrix} \begin{matrix} R_1 \\ R_2 \end{matrix}$$

$$= \det \begin{bmatrix} e^{\alpha x} \cos \beta x & e^{\alpha x} \sin \beta x \\ -\beta e^{\alpha x} \sin \beta x & \beta e^{\alpha x} \cos \beta x \end{bmatrix} \leftarrow R_2 - \alpha R_1$$

$$= e^{\alpha x} \cdot (\beta e^{\alpha x}) \det \begin{bmatrix} \cos \beta x & \sin \beta x \\ -\sin \beta x & \cos \beta x \end{bmatrix}$$

$$\underbrace{\hspace{10em}}_{\cos^2 \beta x + \sin^2 \beta x \equiv 1}$$

$$= \beta e^{2\alpha x} \leftarrow \text{not zero, never zero.}$$

Yes, they give gen^l solⁿ.

Shortcut trick: Just need to see y_1, y_2 are lin independent; $\frac{y_2}{y_1} \leftarrow \text{not constant}$

Check $\frac{e^{\alpha x} \sin \beta x}{e^{\alpha x} \cos \beta x} = \tan \beta x \leftarrow \text{not const.}$ ✓

Euler: Got real gen^l solⁿ! Forget how!

Hmmm; Picked $\alpha + \beta i$ from $\alpha \pm \beta i$. What if I used

$$e^{(\alpha - \beta i)x} = e^{\alpha x} e^{-i\beta x} = e^{\alpha x} \left(\underbrace{\cos(-\beta x)}_{\cos \beta x} + i \underbrace{\sin(-\beta x)}_{-\sin \beta x} \right)$$

$$= \left(\underbrace{e^{\alpha x} \cos \beta x}_{y_1 \text{ same as before}} \right) + i \left(\underbrace{-e^{\alpha x} \sin \beta x}_{\tilde{y}_2 = -y_2} \right)$$

Get same gen^l solⁿ : $y = c_1 y_1 + c_2 \tilde{y}_2 = c_1 y_1 + \underbrace{(-c_2)}_{\substack{\uparrow \\ k_2}} y_2$

$\uparrow$ k_1

Think : As c_1, c_2 range over all real #'s,

k_1, k_2 do the same. ✓

Forget about second root too!

Higher order eqns $ay''' + by'' + cy' + dy = \underline{\underline{0}}$

IVP $\left\{ \begin{array}{l} y(x_0) = y_0 \\ y'(x_0) = y_0' \\ y''(x_0) = y_0'' \end{array} \right.$

Play same game : $L[e^{rx}] = \underbrace{(ar^3 + br^2 + cr + d)}_{=0} e^{rx}$

Get 3 roots : r_1, r_2, r_3

Get $y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + c_3 e^{r_3 x}$

IVP

$$y(x_0) = c_1 e^{r_1 x_0} + c_2 e^{r_2 x_0} + c_3 e^{r_3 x_0} = A$$

$$y'(x_0) = c_1 r_1 e^{r_1 x_0} + c_2 r_2 e^{r_2 x_0} + c_3 r_3 e^{r_3 x_0} = B$$

$$y''(x_0) = c_1 r_1^2 e^{r_1 x_0} + c_2 r_2^2 e^{r_2 x_0} + c_3 r_3^2 e^{r_3 x_0} = C$$

Need $\det \begin{bmatrix} y_1(x_0) & y_2(x_0) & y_3(x_0) \\ y_1'(x_0) & y_2'(x_0) & y_3'(x_0) \\ y_1''(x_0) & y_2''(x_0) & y_3''(x_0) \end{bmatrix} \neq 0$

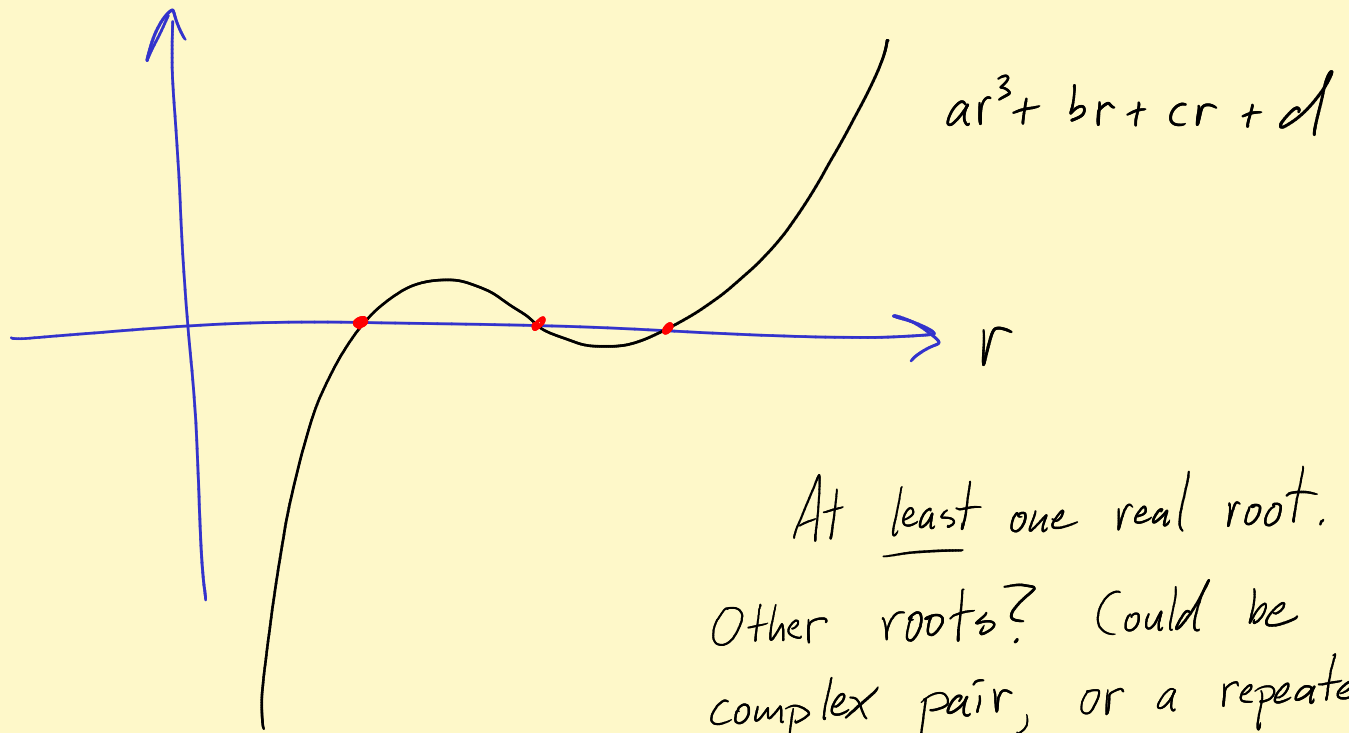
Theory all the same!

Need 3 linearly independent solⁿs.

y_1, y_2, y_3 are lin indep

means: If $c_1 y_1 + c_2 y_2 + c_3 y_3 \equiv 0$,
then all c 's must be zero.

Lin dependent means: One of y 's is a
linear combo of the other y 's



At least one real root.

Other roots? Could be complex pair, or a repeated real root, or all three real and $\neq$, or all 3 real same.