

$y_1, y_2, \dots, y_N$ are linearly independent on (a, b) means:

$$\left[\begin{array}{l} c_1 y_1 + c_2 y_2 + \dots + c_N y_N \equiv 0 \text{ on } (a, b) \\ \text{is only possible if all the } c\text{'s} = 0. \end{array} \right.$$

Note: Not linearly independent is called linearly dependent.

Happens exactly when one of y 's is a linear combination of the others.

EX Are $x^2 + x + 1$, $x^2 + x$, x^2 lin. ind. on $(-\infty, \infty)$?

What if $c_1(x^2 + x + 1) + c_2(x^2 + x) + c_3 x^2 \equiv 0$ on $\mathbb{R}$?

$$\underbrace{(c_1 + c_2 + c_3)}_{\text{must} = 0} x^2 + \underbrace{(c_1 + c_2)}_{=0} x + \underbrace{(c_1)}_{=0} \equiv 0$$

$$\#3 : c_3 = 0$$

$$\#2 : c_2 = 0$$

$$\#1 : c_1 = 0$$

All c 's must be zero. Lin. ind. ✓

EX What about $\sin^2 x$, $\cos 2x$, 3 ?

$$\text{Trig identity: } \sin^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin^2 x - \frac{1}{2} \cos 2x - \left(\frac{1}{2 \cdot 3}\right) 3 \equiv 0$$

Lin. Dep. ✓

Prob

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y \equiv 0$$

↑
homog

Try $y = e^{rx}$.

$$\underbrace{[a_n r^n + \dots + a_1 r + a_0]}_{\text{Char. eqn.} = 0} e^{rx} \equiv 0$$

↑ never zero ↑ want

Need n lin. ind. solⁿs. Good news: n -th degree polys have n roots. [Fund. Thm. of Algebra: They have at least one root in $\mathbb{C}$.]

Best case Get n real and distinct roots

$$r_1, r_2, r_3, \dots, r_n.$$

$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_n e^{r_n x}$ is the gen^l solⁿ.

IVP: Need n -conditions to pin down c 's.

$$n \left\{ \begin{array}{l} y(x_0) = y_0 \\ y'(x_0) = y_0' \\ \vdots \\ y^{(n-1)}(x_0) = y_0^{(n-1)} \end{array} \right.$$

Write out. Make a matrix linear sys.

Wronskian = $\det [n \times n]$

$$W = \det \begin{bmatrix} e^{r_1 x} & \dots & e^{r_n x} \\ r_1 e^{r_1 x} & \dots & r_n e^{r_n x} \\ \vdots & \ddots & \vdots \\ r_1^{n-1} e^{r_1 x} & \dots & r_n^{n-1} e^{r_n x} \end{bmatrix}_{n \times n}$$

never zero. Got gen^l solⁿ.

Factoring polys with real coeff

EX

$$r(r-2)(r-3)^3(r^2+2r+2)^2(r^2+1)=0$$

$\uparrow$ $r=0$ $e^{0 \cdot x} = 1$
 $\uparrow$ $r=2$ e^{2x}
 $\uparrow$ $r=3,3,3$ $e^{3x}, xe^{3x}, x^2 e^{3x}$
 $\uparrow$ r^2+2r+2 $r = -1 \pm i$ repeated
 $\uparrow$ r^2+1 $r = \pm i$
 $e^{0 \cdot x} \cos x, e^{0 \cdot x} \sin x$
 $\cos x, \sin x$

4 solⁿ $\left[\begin{array}{l} e^{-x} \cos x, e^{-x} \sin x \leftarrow \text{once} \\ x e^{-x} \cos x, x e^{-x} \sin x \leftarrow \text{repeated} \end{array} \right.$

Fact Polys with real coeff factor into terms

1) $(r-r_0)^m$ r_0 real. m = multiplicity

EX: $r = (r-0)^1$
 $(r-7)$
 $(r-1)^{13}$

2) quadratic polys with real coeff with a pair of complex roots $\alpha \pm \beta i$.

$$(r^2 + br + c)^m \quad b^2 - 4c < 0$$

EX $(r^2 + 1)$
 $(r^2 + 2r + 2)^{10}$

Simple root case: $(r - r_0) \quad e^{r_0 x} \quad \text{sol}^n$

Repeated real root: $(r - r_0)^m$

Get

$$\left. \begin{array}{l} e^{r_0 x} \\ x e^{r_0 x} \\ x^2 e^{r_0 x} \\ \vdots \\ x^{m-1} e^{r_0 x} \end{array} \right\} m \text{ solutions!}$$

Euler trick

$$L[e^{rx}] = (r - r_0)^m Q(r) e^{rx}$$

$$\text{Diff: } \frac{2}{2r}$$

$$\frac{2}{2r} L[e^{rx}] = \text{vanishes at } r_0 \text{ to order } m$$

$$L\left[\frac{2}{2r} e^{rx}\right] \quad \text{Let } r = r_0$$

$$x e^{r_0 x} \quad r = r_0 \quad x e^{r_0 x}$$

Can repeat m times!

Complex simple roots $(r^2 + br + c)$ roots $\alpha \pm \beta i$

Get $e^{\alpha x} \cos \beta x, \quad e^{\alpha x} \sin \beta x$

Note: If $\alpha = 0$, Get $\cos \beta x, \sin \beta x$

Complex roots with multiplicity m

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$$(r^2 + br + c)^m$$

$$\alpha \pm \beta i$$

Take $\alpha + \beta i$

Complex solⁿs

$$\left. \begin{array}{l} e^{(\alpha + \beta i)x} \\ x e^{(\alpha + \beta i)x} \\ x^2 e^{(\alpha + \beta i)x} \\ \vdots \\ x^{m-1} e^{(\alpha + \beta i)x} \end{array} \right\} m \text{ complex sol}^n\text{s}$$

$$e^{\alpha x} \cos \beta x, e^{\alpha x} \sin \beta x$$

$$x e^{\alpha x} \cos \beta x, x e^{\alpha x} \sin \beta x$$

 $\vdots$

$$x^{m-1} e^{\alpha x} \cos \beta x, x^{m-1} e^{\alpha x} \sin \beta x$$

 $2m$ real valued solⁿs.Finally, Get n real solⁿs.

Factoring n -th degree polys with real coeff's

$$\text{EX } r^4 + 8r^2 + 16 = 0$$

$$y^{(4)} + 8y'' + 16y = 0$$

$$(r^2)^2 + 8(r^2) + 16 = 0$$

$$\left(\underbrace{r^2 + 4} \right)^2 = 0$$

$$r = \pm 2i, \pm 2i \leftarrow \text{repeated}$$

$$\cos 2x, \sin 2x$$

$$(\alpha = 0)$$

$$x \cos 2x, x \sin 2x$$

Genl solⁿ $y = c_1 \cos 2x + c_2 \sin 2x + c_3 x \cos 2x + c_4 x \sin 2x$

EX $r^6 + 5r^5 + 4r^4 = 0$

$$r^4 (r^2 + 5r + 4) = 0$$

$$r^4 (r+1)(r+4) = 0$$

$$r = 0, 0, 0, 0$$

$$e^{0 \cdot x}, x e^{0 \cdot x}, x^2 e^{0 \cdot x}, x^3 e^{0 \cdot x}$$

$$1, x, x^2, x^3$$

$$e^{-x} \quad e^{-4x}$$

$$y = \underline{c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^{-x} + c_6 e^{-4x}}$$

Fact If a poly $P(r)$ vanishes at r_0

$[P(r_0) = 0]$, then $(r - r_0)$ divides $P(r)$.

EX $r^3 - r^2 + r - 1 = P(r)$. See $P(1) = 0$

So $r - 1$ divides $P(r)$.

$$\begin{array}{r}
 r^2 + 1 \\
 r-1 \overline{) r^3 - r^2 + r - 1} \\
 \underline{r^3 - r^2} \\
 r-1 \\
 \underline{r-1} \\
 0 \leftarrow \text{no remainder}
 \end{array}$$

So $P(r) = (r-1)(r^2+1)$

Operator notation

$$D = \frac{d}{dx}$$

$$\begin{aligned}
 ay'' + by' + cy &= aD^2y + bDy + cy \\
 &= [aD^2 + bD + c]y
 \end{aligned}$$

Factor poly $\sim$ factoring operator

$$D^2 + 5D + 6 = (D+3)(D+2)$$