

Lecture 17 Vibrations (3.4)

MyLab HW 16

1

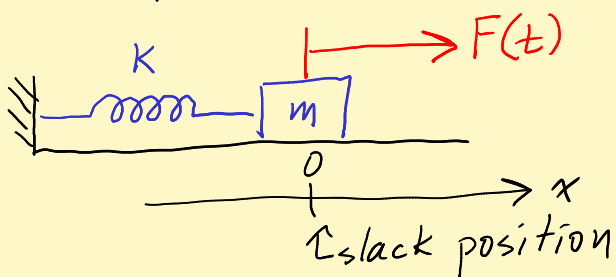
Exam 1: Thurs. Oct 8, 8:00-9:00 pm in WTHR 200 (Main)

Our section: 266-116 MWF 1:30 pm

Exam 1 covers Lessons 1-16 (up to 3.3). Ten problems: prob 1-7 multiple choice, 8-10 written answer. No scantrons. closed book, no notes or crib sheets, no electronics. Darken multiple choice bubbles in exam booklet. Bring pencils and erasers. Papers will be graded in GS. Only front side of pages will be scanned. Back sides can be used for scratch. Seating chart, old exam archive, and Practice problems at www.math.purdue.edu/MA266

Practice probs 1-16 make a good review.

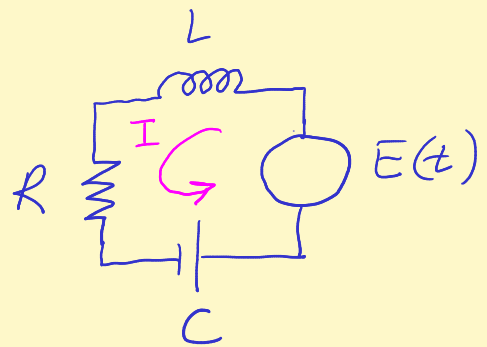
Monday: Review lecture.



$$F_{\text{net}} = ma$$

$$F(t) - Kx - c\dot{x} = m\ddot{x}$$

$$\underline{m\ddot{x} + c\dot{x} + Kx = F(t)}$$



Kirchoff's law

$$L \frac{dI}{dt} + RI + \frac{1}{C}Q + (-E(t)) = 0$$

$$\underline{L \frac{d^2I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C}I = E'(t)}$$

$$m \ddot{y} + c \dot{y} + k y = 0 \leftarrow \text{no external force}$$

Case no friction. $\boxed{c=0}$

$$m \ddot{y} + k y = 0$$

$$m r^2 + k = 0$$

$$r = \sqrt{\frac{k}{m}} i$$

$$y = c_1 \cos \sqrt{\frac{k}{m}} t + c_2 \sin \sqrt{\frac{k}{m}} t$$

$\omega_0 \leftarrow$ natural angular freq of motion

IVP

$$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

do this
before

$$\rightarrow y' = -c_1 \omega_0 \sin \omega_0 t + c_2 \omega_0 \cos \omega_0 t$$

$$y(t_0) = c_1 \cos \omega_0 t_0 + c_2 \sin \omega_0 t_0 = A$$

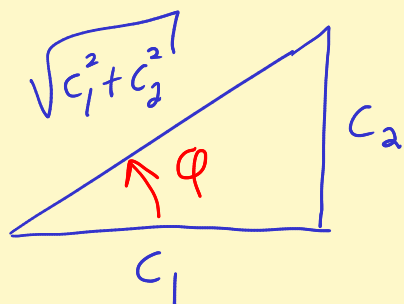
$$\rightarrow y'(t_0) = -c_1 \omega_0 \sin \omega_0 t_0 + c_2 \omega_0 \cos \omega_0 t_0 = B$$

Use Cramer's rule to get c 's.

Get ans: Wiggling $\cos$ and $\sin$. Messy?

Classic important trick

$$y = \underbrace{\sqrt{c_1^2 + c_2^2}}_A \left(\underbrace{\frac{c_1}{\sqrt{c_1^2 + c_2^2}}}_{\cos \varphi} \cos \omega_0 t + \underbrace{\frac{c_2}{\sqrt{c_1^2 + c_2^2}}}_{\sin \varphi} \sin \omega_0 t \right)$$



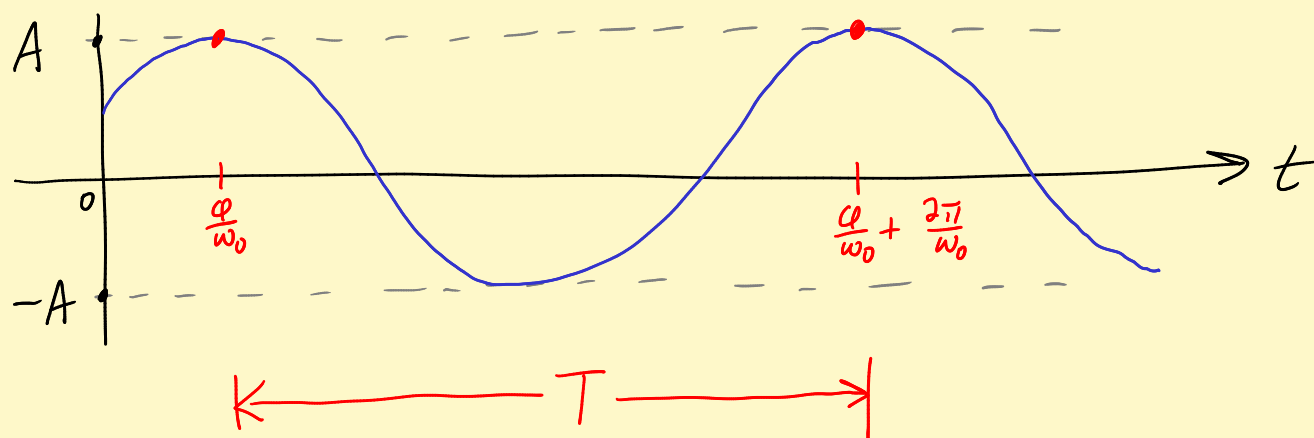
$$A = \sqrt{c_1^2 + c_2^2}$$

$$\varphi = \tan^{-1} \frac{c_2}{c_1} \leftarrow \text{from } \Delta$$

$$= A \cos(\omega_0 t - \varphi) \leftarrow \text{easy to graph!}$$

φ = "phase shift"

A = "amplitude"



$$T = \text{"period"} = \frac{2\pi}{\omega_0}$$

$$\text{Frequency: } f = \frac{1}{T} = \frac{\omega_0}{2\pi}$$

Trig identities: $e^{(\alpha+\beta)i} = e^{\alpha i} \cdot e^{\beta i} = (\cos \alpha + i \sin \alpha)(\cos \beta + i \sin \beta)$

$$\cos(\alpha+\beta) + i \sin(\alpha+\beta) = \left[\cos\alpha \cos\beta - \sin\alpha \sin\beta \right] + \left[\cos\alpha \sin\beta + \sin\alpha \cos\beta \right] i$$

$\uparrow$ $i^2 = -1$

Damping. $c > 0$

$$m \ddot{y} + c \dot{y} + ky = 0$$

$$mr^2 + cr + k = 0$$

roots: $r_1, r_2 = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$

Case r_1, r_2 real and unequal. $c^2 - 4km > 0$

Overdamped

$$r_1 = \frac{-c}{2m} - \frac{\sqrt{c^2 - 4km}}{2m} \quad \leftarrow \text{negative } \checkmark$$

$$r_2 = \frac{-c}{2m} + \frac{\sqrt{c^2 - 4km}}{2m} \quad \leftarrow \text{also negative!}$$

Hmmm.

$$c^2 > c^2 - 4km > 0$$

$\leftarrow$ our case

So

$$c > \sqrt{c^2 - 4km} > 0$$

Divide by $2m$:

$$\frac{c}{2m} > \frac{\sqrt{c^2 - 4km}}{2m} > 0$$

Subtract $-\frac{c}{2m}$: Get

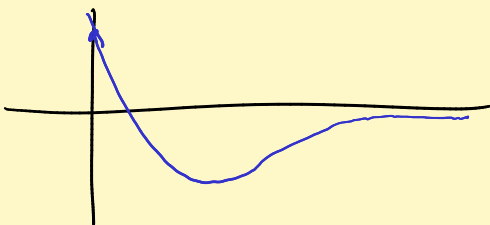
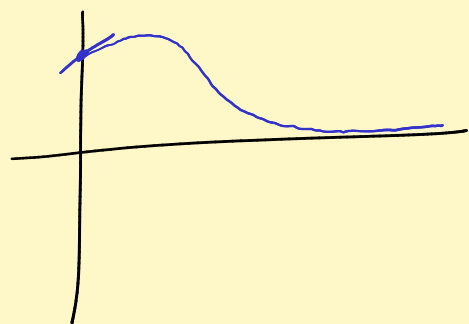
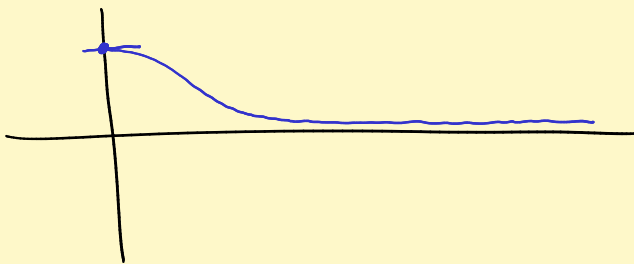
$$0 > \underbrace{-\frac{c}{2m} + \frac{\sqrt{c^2 - 4km}}{2m}}_{r_2} > -\frac{c}{m}$$

r_2 is < 0 too. ✓

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} \quad \text{both } r\text{'s} < 0.$$

Dies away fast!

Fun fact It "wiggles" at most once.



At most one "hump".

Crosses x-axis at most once,

Fun calculus.

Critically damped Two real roots, the same

$$\underline{c^2 - 4km = 0}$$

$$r_1, r_2 = \frac{-c}{2m}$$

$$y = c_1 e^{-\frac{c}{2m}t} + c_2 t e^{-\frac{c}{2m}t}$$

also $\rightarrow 0$ very fast as $t \rightarrow \infty$.

Same: At most one hump. At most one crossing.

Delicate! lighter m : Overdamped - no wiggles
heavier m : Complex roots - wiggling.

Underdamped $r_1, r_2 = \frac{-c}{2m} \pm \underbrace{\frac{\sqrt{|c^2 - 4km|}}{2m}}_{\omega} i$

$$\underline{c^2 - 4km < 0}$$

$$y = c_1 e^{-\frac{c}{2m}t} \cos \omega t + c_2 e^{-\frac{c}{2m}t} \sin \omega t$$

$$= A e^{-\frac{c}{2m}t} \cos(\omega t - \phi)$$

$\uparrow$ goes to zero fast

