

Lecture 18 The Method of Undetermined Coefficients

Exam 1: Tomorrow 10/8 at 8:00 pm in WTHR 200 (main), No class Fri.

MyLab 18
due Wed after break
GS HW 16, 17, 18

Fact $A(x)y'' + B(x)y' + C(x)y = F(x)$ $(+)$ $\leftarrow$ non-homogeneous

$$A(x)y'' + B(x)y' + C(x)y = 0 \quad (*) \leftarrow \text{homogeneous}$$

Gen^l solⁿ to $(+)$ is $y = \underbrace{c_1 y_1 + c_2 y_2}_{\substack{\text{Gen}^l \text{ sol}^n \text{ to } (*) \\ y_c}} + y_p$ $\uparrow$ one "particular solⁿ" to $(+)$

Why $L[y] = Ay'' + By' + Cy \leftarrow$ linear operator

Suppose y_p solves $(+)$: $L[y_p] = F$

Any other solⁿ $\underline{Y}$ to $(+)$: $L[\underline{Y}] = F$

$$L[\underline{Y} - y_p] = \underbrace{F - F}_0$$

bottom - top,
 L is linear

Aha! $\underline{Y} - y_p$ solves $(*)$. So there are c_1 and c_2 such that

$$\underline{Y} - y_p = c_1 y_1 + c_2 y_2$$

$$\underline{Y} = (c_1 y_1 + c_2 y_2) + y_p \quad \checkmark$$

Also check thing like this solve $(+)$:

Easy : $L[c_1 y_1 + c_2 y_2 + y_p] = c_1 \underbrace{L[y_1]}_0 + c_2 \underbrace{L[y_2]}_0 + \underbrace{L[y_p]}_F$

EX $y'' + 4y = e^{-2x}$

Step 1 Solve homog $y'' + 4y = 0 \quad (*)$

$$r^2 + 4 = 0$$

$$r = \pm 2i$$

2

$$y_c = c_1 \cos 2x + c_2 \sin 4x$$

Step 2 Find one stupid y_p .

Big idea: Guess $y_p = A e^{-2x}$

$$y_p' = (-2) A e^{-2x}$$

$$y_p'' = (-2)^2 A e^{-2x}$$

Plug in $\frac{1}{8}$ force!

$$y_p'' + 4 y_p = e^{-2x} \quad \text{want}$$

$$\underbrace{4Ae^{-2x} + 4Ae^{-2x}}_{\substack{\text{need} = 1}} = e^{-2x}$$

$$\boxed{A = 1/8}$$

$$y_p = \frac{1}{8} e^{-2x}$$

Step 4 Gen^l solⁿ to (t) is $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8} e^{-2x}$

Method of Undet. coeff

$$\underbrace{a y'' + b y' + c y}_{\text{const. coeff}} = \underbrace{F(x)}_{\text{special}}$$

$F(x)$	Try this "trial sol"
e^{rx}	Ae^{rx}
x^n or any poly of deg n .	$A_n x^n + A_{n-1} x^{n-1} + \dots + A_1 x + A_0$ (gen ^l poly of deg n)
$\cos wx$ $\sin wx$ $a \cos wx + b \sin wx$	$A \cos wx + B \sin wx$
$e^{rx} \cos wx$ $e^{rx} \sin wx$	$Ae^{rx} \cos wx + Be^{rx} \sin wx$
$x^n e^{rx} \cos wx$ $x^n e^{rx} \sin wx$	$(A_n x^n + \dots + A_1 x + A_0) e^{rx} \cos wx +$ $(B_n x^n + \dots + B_1 x + B_0) e^{rx} \sin wx$

Rule But, if any single piece of the trial solⁿ from the table solves the homog prob, then the correct try is

$$y_p = x^s \left(\text{trial sol}^n \text{ from table} \right)$$

where s is the smallest positive integer to make
no single piece of this guess solves homog.

EX $y'' + y = \cos x \leftarrow \text{resonance!}$

homog solⁿ : $r^2 + 1 = 0 \quad r = \pm i$

$y_c = c_1 \cos x + c_2 \sin x$

Table: Trial solⁿ : $\underbrace{A \cos x}_{\uparrow \text{pieces}} + \underbrace{B \sin x}_{\uparrow}$ $\leftarrow \text{solves homog}$

Correct try : $x^s (A \cos x + B \sin x)$

$Ax^s \cos x + Bx^s \sin x$
 $s=1$ works. $\underbrace{\quad}_{\text{pieces}} \quad \uparrow \quad \uparrow$ $x \cos x, x \sin x$
do not solve homog

Try $y_p = \underbrace{x}_{u} \underbrace{(A \cos x + B \sin x)}_v$

$y_p' = (A \cos x + B \sin x) + x(-A \sin x + B \cos x)$

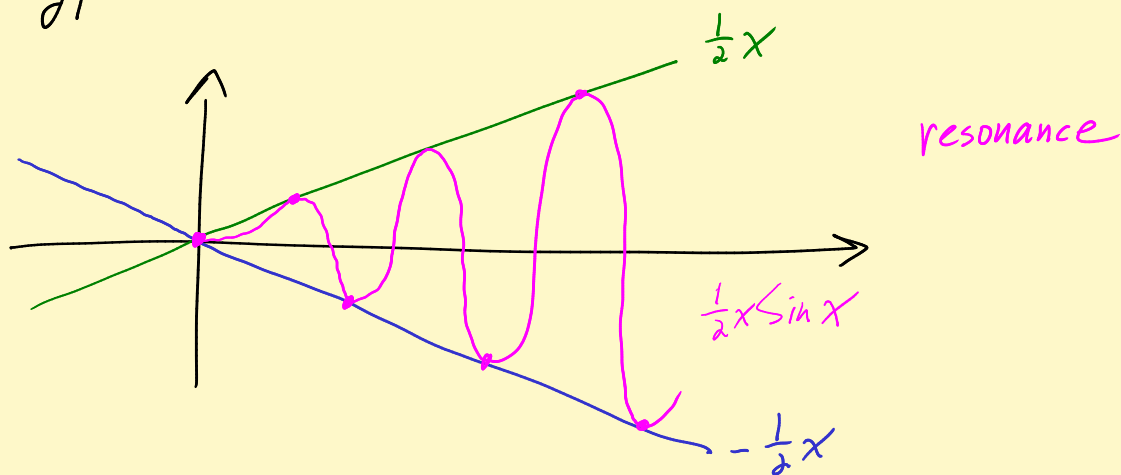
$y_p'' = \underbrace{0}_{u''} \cdot \underbrace{(A \cos x + B \sin x)}_v + 2 \cdot \underbrace{1}_{u'} \cdot \underbrace{[-A \sin x + B \cos x]}_{v'} + \underbrace{x}_{u} \cdot \underbrace{[-A \cos x - B \sin x]}_{v''}$
 $[uv]'' = u''v + 2u'v' + uv''$

Plug into (+). Collect terms

Miracle! $-2A \sin x + 2B \cos x = \cos x \leftarrow \text{want}$

Get. Need $-2A = 0$, $2B = 1$
 $A = 0$, $B = 1/2$

Ans: $y_p = \frac{1}{2}x \sin x$



EX $y'' + 4y' + 4y = e^{-2x}$

homog: $r^2 + 4r + 4 = 0$
 $(r+2)^2 = 0$ $r = -2, -2$

$$y_c = c_1 e^{-2x} + c_2 x e^{-2x}$$

Trial solⁿ from table: Ae^{-2x} ← ouch! solves (*)
 $x^s(Ae^{-2x})$ $s=1$ Axe^{-2x} ← ouch!
 $s=2$ Ax^2e^{-2x} ← safe!

Correct try $y_p = Ax^2e^{-2x}$

Fun thing

$$(D^2 + 4D + 4)y = e^{-2x}$$

$$(D+2)^2 y = e^{-2x}$$

What diff operator "annihilates" RHS e^{-2x} ?

Answer: $(D+2)$ does

$$(D+2) \left[(D+2)^2 y \right] = (D+2)e^{-2x} = 0$$

$$(D+2)^3 y = 0$$

$$y = \underbrace{c_1 e^{-2x} + c_2 x e^{-2x}}_{\text{homog sol}^n} + \underbrace{c_3 x^3 e^{-2x}}_{y_p \text{ that we need.}}$$

HWK prob

$$y^{(5)} + 5y^{(4)} - y = 17 \leftarrow \text{poly of deg 0}$$

Easy to find y_p without solving nasty 5-th deg thing.

Table says try $y_p = \text{poly of deg zero} = A$.

It works! Don't need to go any further.

Meaning of "pieces of" a trial solⁿ:

EX

$$(A_1 x + A_0) e^{2x} \cos 3x + (B_1 x + B_0) e^{2x} \sin 3x$$

$$\underbrace{A_1 x e^{2x} \cos 3x}_{\uparrow} + \underbrace{A_0 e^{2x} \cos 3x}_{\uparrow} + \underbrace{B_1 x e^{2x} \sin 3x}_{\uparrow} + \underbrace{B_0 e^{2x} \sin 3x}_{\uparrow}$$

Four "pieces"