Lesson 3 Linear First Order ODE

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EX: $$\left(e^{x^2}+\sqrt{x}\right)\frac{dy}{dx}+\left(\tan x+x^2\right)y=\left(\sin\sqrt{x}+x\right)$$

EX: First order linear with constant coeff.

$$ay'+by=c$$

$$y'=\frac{c-by}{a}$$

$$\underbrace{\frac{a}{c-by}}\,y'=1$$

Chain rule!

$$\frac{d}{dy}\left[\frac{-a}{b}\ln|c-by|\right]$$

$$\frac{d}{dx}\left[-\frac{a}{b}\ln|c-by|\right]=1$$

Integrate: $-\frac{a}{b}\ln|c-by|=x+K$, solve for y.

EX: $$\underbrace{x^2\frac{dy}{dx}+2x\,y}_{\frac{d}{dx}\left[x^2y\right]}=e^x$$

$$x^2y=\int e^x\,dx=e^x+C$$

$$\boxed{y=\frac{1}{x^2}e^x+\frac{C}{x^2}}$$

Ouch! Blows up at $x=0$.

Why blow up: $\frac{dy}{dx} = \frac{e^x - 2xy}{x^2}$ ⟵ nasty at $x = 0$.

Zeroes of coeff in front of $\frac{dy}{dx}$ cause bad behavior.

EX: Draw a direction field for $y' + 7y = t + e^{-6t}$

$$\frac{dy}{dt} = \underbrace{-7y + t + e^{-6t}}_{f(t,y)}$$

How to solve linear 1st order ODE:

$$A(x)\frac{dy}{dx} + B(x)\,y = C(x)$$

Step 1: Put in standard form:

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

where $P = \frac{B(x)}{A(x)}$

$Q = \frac{C(x)}{A(x)}$

zeroes of $A(x)$ are bad!

Step 2: Big idea: Multipy by an "integrating factor" $u(x)$:

$$\underbrace{u\frac{dy}{dx} + uP\,y}_{\text{wish this was the derivative of } uy} = uQ$$

Hmmm: $\frac{d}{dx}[uy] = u\frac{dy}{dx} + \frac{du}{dx}\,y$

Aha! Pick u so that $\frac{du}{dx} = uP$

$$\underbrace{\frac{1}{u}\frac{du}{dx}} = P$$

Chain rule $\qquad \frac{d}{dx}\ln|u| = P$

$$e^{\ln|u|} = e^{\int P\,dx + C}$$

$$|u| = e^{\int P\,dx}$$ ← Take $C=0$. Just want one.

$$u = \pm e^{\int P\,dx}$$ ← Take the + one.

$$\boxed{u = e^{\int P\,dx}}$$

Step 3: Multiply by $u = e^{\int P\,dx}$ and finish up:

$$\underbrace{u\frac{dy}{dx} + uPy}_{\frac{d}{dx}[uy]} = uQ$$

So $\qquad uy = \int uQ\,dx + C$

Solve for y: $\qquad y = \frac{1}{u}\int uQ\,dx + \frac{C}{u}$

↑ don't memorize!

Memorize:

1. Standard Form.
2. $u = e^{\int P\,dx}$
3. Mult. Standard Form eqn by u and do the routine.

EX: $\frac{1}{3}\frac{dy}{dx} + y = \frac{1}{3}e^{2x}$

Step 1: Mult by 3:

$P(x) = 3$

$$(*)\quad \frac{dy}{dx} + 3y = e^{2x}$$

Step 2: Find $u = e^{\int P\,dx} = e^{\int 3\,dx} = e^{3x}$

Forget about $+C$, Just want one.

Step 3: Multiply S.F. eqn by u

$$\underbrace{e^{3x}\left(\frac{dy}{dx} + 3y\right)}_{\frac{d}{dx}\left[e^{3x}y\right]} = e^{3x}\cdot e^{2x}$$

$$\frac{d}{dx}\left[e^{3x}y\right] = e^{5x}$$

So $e^{3x}y = \int e^{5x}\,dx = \frac{1}{5}e^{5x} + C$

$$\boxed{y = \frac{1}{5}e^{2x} + Ce^{-3x}}$$

Gen^{l} Sol^{n}

Initial condition: Find the solution with $y(0) = 2$.

want

$$y(0) = \frac{1}{5}e^{2\cdot 0} + Ce^{-3\cdot 0} = 2$$

$$\frac{1}{5} + C = 2$$

$$\boxed{C = 9/5}$$

$$y = \frac{1}{5}e^{2x} + \frac{9}{5}e^{-3x}$$

EX: $y' - (\operatorname{Tan} x)\, y = 8 \operatorname{Sin}^3 x$

1 here, Standard Form ✓

So $u = e^{\int P\,dx} = e^{\int -\operatorname{Tan} x \, dx}$

$= e^{-(\operatorname{Ln}|\sec x|)}$ $\qquad \operatorname{Ln} a^b = b \operatorname{Ln} a$

$= e^{\operatorname{Ln}|\sec x|^{-1}} = |\sec x|^{-1} = \left|\frac{1}{\cos x}\right|^{-1}$

$= |\cos x| = \pm \cos x$ ← pick the + one.

Get $\boxed{u = \cos x}$

Use it: $\underbrace{\cos x\left(\underbrace{\frac{dy}{dx} - (\operatorname{Tan} x)\, y}_{\text{S.Form!}}\right)}_{\frac{d}{dx}\left[(\cos x)\, y\right]} = \cos x \, 8 \operatorname{Sin}^3 x$

So $(\cos x)\, y = 8 \int \underset{\uparrow\, u = \sin x}{\operatorname{Sin}^3 x} \underbrace{\cos x \, dx}_{du} = 8 \frac{\operatorname{Sin}^4 x}{4} + C$

$y = 2 \frac{\operatorname{Sin}^4 x}{\cos x} + \frac{C}{\cos x}$

$$\boxed{y = 2 \operatorname{Tan} x \operatorname{Sin}^3 x + C \sec x}$$

Formula can be useful:

$$\int e^{-x^2}\,dx = ?$$

$$\int e^{-x^2}\,dx = \int_0^x e^{-s^2}\,ds + C$$

Integrate from initial condition point.

Hint L'H Rule: $\frac{d}{dx}\int_0^x e^{-s^2}\,ds = e^{-x^2}$