Lesson 4 Separable Eqns

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

$$\int g(y)\,dy = \int f(x)\,dx$$

$$G(y) = F(x) + C$$ ← defines $y(x)$ implicity

Solve for y if you can.

EX: $\frac{dy}{dx} = xy = \frac{x}{\left(\frac{1}{y}\right)}$ ✓

$$\int \frac{1}{y}\,dy = \int x\,dx$$

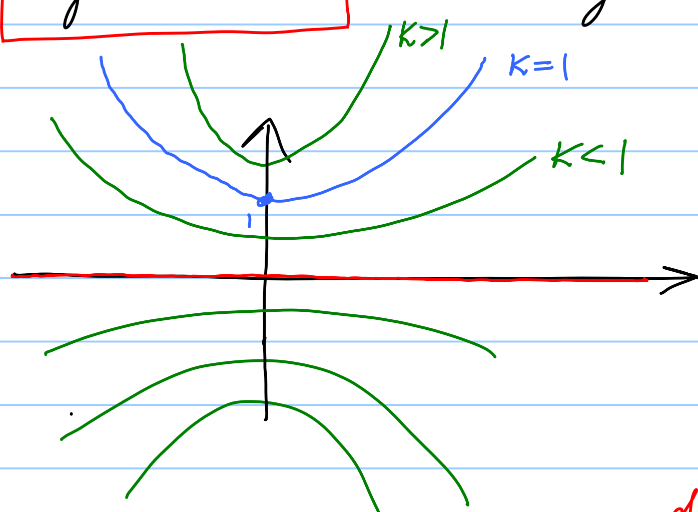
$$e^{\ln|y|} = e^{\frac{1}{2}x^2 + C}$$

$$|y| = e^{\frac{1}{2}x^2 + C} = e^{\frac{1}{2}x^2} e^{C}$$

$$y = \underbrace{\pm e^{C}}_{K \neq 0} e^{\frac{1}{2}x^2}$$

$$\boxed{y = Ke^{\frac{1}{2}x^2}}$$

$y(0) = K$



$K=0$ Hmmm.

Is $y \equiv 0$ a solution?

$\frac{dy}{dx} = xy$ yes!

EX: $\frac{dy}{dx} = e^{x+y} = e^x e^y$

$$\int \frac{1}{e^y}\, dy = \int e^x \, dx$$

$$\int e^{-y}\, dy = \int e^x \, dx$$

$$-e^{-y} = e^x + C$$

$$e^{-y} = -C - e^x$$

$$\underbrace{\ln e^{-y}}_{-y} = \ln(-C - e^x)$$

$$y = -\ln(-C - e^x)$$ ← C better be < 0, $C = -K$. $(K > 0)$

$$\boxed{y = -\ln(K - e^x)}$$ ← boom! when $e^x = K$, $x = \ln K$

Advice: If an ODE is both linear and separable, Go with linear method.

EX: $A\frac{dy}{dt} + By = C$

Step 1: Put it in Standard Form. Divide by A:

$$\frac{dy}{dt} + \underbrace{\frac{B}{A}}_{a} y = \underbrace{\frac{C}{A}}_{b}$$

$$(*) \quad \boxed{\frac{dy}{dt} + ay = b}$$

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x) \qquad u = e^{\int P(x)\,dx}}$$

Step 2: Integrating factor: $u = e^{\int a\,dt} = e^{at}$

no $+C$ here. Just need one u

Step 3: $\underbrace{e^{at}\left(\frac{dy}{dt} + ay\right)}_{\frac{d}{dt}\left[e^{at} y\right]} = be^{at}$

Step 4: So $e^{at} y = \int b e^{at}\,dt = \frac{b}{a} e^{at} + C$

Step 5: $\boxed{y = \frac{b}{a} + C e^{-at}}$

EX: $\frac{dy}{dx} = \frac{xy(y+1)}{(x+1)}$

$$\int \frac{1}{y(y+1)}\,dy = \int \frac{x}{x+1}\,dx$$

$$\frac{x}{x+1} = \frac{x+1-1}{x+1} = 1 - \frac{1}{x+1}$$

Partial Fractions: $\frac{1}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1}$

Step 1: Multiply by denom of LHS:

$$1 = A(y+1) + By$$

Step 2: Collect powers:

$$0 \cdot y + 1 = (A+B)\,y + (A)$$

Step 3: Equate coeff.

$$A+B=0 \quad \leftarrow \quad B=-1$$
$$A=1 \quad \leftarrow \quad A=1$$

So $$\frac{1}{y(y+1)} = \frac{1}{y} - \frac{1}{y+1}$$

$$\int \frac{1}{y(y+1)}\,dy = \int \frac{x}{x+1}\,dx$$

$$\int \frac{1}{y} - \frac{1}{y+1}\,dy = \int 1 - \frac{1}{x+1}\,dx$$

$$\ln|y| - \ln|y+1| = \underbrace{x - \ln|x+1| + C}_{H(x)} \quad \leftarrow \text{Ouch!}$$

$$\ln\frac{|y|}{|y+1|} = H(x)$$

$$\left|\frac{y}{y+1}\right| = e^{H(x)}$$

$$\frac{y^{+1-1}}{y+1} = \underbrace{\pm e^{H(x)}}_{A}$$

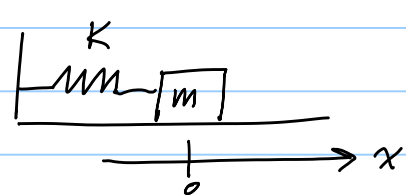
$$1 - \frac{1}{y+1} = \pm e^{H(x)}$$

$$y = A(y+1) = Ay + A$$
$$y(1-A) = A$$
$$y = \frac{A}{1-A}$$

Solve for y:

$$\boxed{y = -1 + \frac{1}{1 \pm e^{x - \ln|x+1| + C}}} \quad \leftarrow \text{as } x \to \infty,\ y(x) \to -1.$$

Famous problems:



0 slack position.

$F = ma$ Newton

$-kx = ma$ Hooke's

D.E. : $-kx = m\dfrac{dv}{dt}$ ← ouch! 3 vars

<u>Big idea</u>: Chain rule: $\dfrac{dv}{dt} = \dfrac{dv}{dx}\underbrace{\dfrac{dx}{dt}}_{v} = v\dfrac{dv}{dx}$

D.E. : $-kx = mv\dfrac{dv}{dx}$ ← Separable!

$$\int -kx\,dx = \int mv\,dv$$

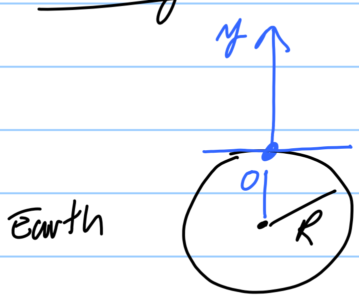
$$-\frac{1}{2}kx^2 = \frac{1}{2}mv^2 + C$$

Say $v = v_0$ and $x = x_0$ when $t = 0$.

Then $-\frac{1}{2}kx_0^2 = \frac{1}{2}mv_0^2 + C$ ← Get C

$$\underbrace{-\left[\frac{1}{2}kx^2 - \frac{1}{2}kx_0^2\right]}_{-\Delta(\text{Potential Energy})} = \underbrace{\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2}_{\Delta(\text{Kinetic Energy})}$$

<u>Coming soon</u>: Shoot a canonball into space



$F = ma$

$$-\frac{GMm}{(y+R)^2} = m\frac{dv}{dt} = m\frac{dv}{dy}\underbrace{\frac{dy}{dt}}_{v}$$

$= mv\dfrac{dv}{dy}$ ← Separable!