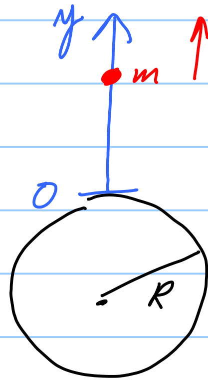
Lesson 5 Separable and homogeneous equations

Written problems for Lessons 4, 5 due next Wed in class

Webassign for Lesson 5 due Mon, 11 pm.



Shoot canonball into space

$$F = ma$$

$$-\frac{GMm}{(y+R)^2} = m\frac{dv}{dt} = m\frac{dv}{dy}\underbrace{\frac{dy}{dt}}_{v} = mv\frac{dv}{dy}$$

$$\int -\frac{GM}{(y+R)^2}\,dy = \int v\,dv$$

$$\frac{GM}{(y+R)} = \frac{1}{2}v^2 + C \longleftarrow$$ Stop! and figure out C.

Assume $v = v_0$ and $y = 0$ when $t = 0$.

$$\frac{GM}{(0+R)} = \frac{1}{2}v_0^2 + C \longleftarrow \text{ So } C = \frac{GM}{R} - \frac{v_0^2}{2}$$

$$(*) \quad \boxed{v^2 = \frac{2GM}{(y+R)} - \frac{2GM}{R} + v_0^2}$$

How high does the ball go?

High point is where $v = 0$.

$$0^2 = \frac{2GM}{(y_{max}+R)} - \frac{2GM}{R} + v_0^2$$

$$\frac{2GM}{(y_{max}+R)} = \frac{2GM}{R} - v_0^2 \longleftarrow$$ If $v_0 < \sqrt{\frac{2GM}{R}}$, the ball turns around.

Aha! If $\boxed{v_0 = \sqrt{\frac{2GM}{R}}}$, the ball goes forever! $\longleftarrow$ Escape velocity!

Solve (*): $\underbrace{v}_{\frac{dy}{dt}} = \sqrt{\dfrac{2GM}{(y+R)} - C}$ Separable!

$$\int \frac{dy}{\sqrt{\dfrac{2GM}{(y+R)} - C}} = \int dt$$

MAPLE:

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> int(1/sqrt(2*G*M/(y+R) - C), y);
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Hard integral! But if $v_0 = \sqrt{\dfrac{2GM}{R}}$, the integral is easy.

Homogeneous eqns: $\dfrac{dy}{dx} = F\left(\dfrac{y}{x}\right)$ ← a fcn that only depends on ratio $\frac{y}{x}$ is called "homogeneous"

Method: Change of variables: $v = \dfrac{y}{x}$ ← v = new dependent var, x is the indep. var. [$y(x)$ is a fcn of x]

Step 1: Change $\dfrac{dy}{dx}$ to something in terms of v:

$v = \dfrac{y}{x}$

So $\boxed{y = xv}$ and $\dfrac{dy}{dx} = \dfrac{d}{dx}[xv] = \underline{1 \cdot v + x\dfrac{dv}{dx}}$

Step 2: Get new ODE in x and v:

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

$$v + x\frac{dv}{dx} = F(v)$$ ← separable!

Step 3: Solve it

$$\int \frac{dv}{F(v)-v} = \int \frac{dx}{x}$$

$$H(v) = \ln|x| + C$$ ← solve for v if possible

Step 4: Plug in y/x for v and solve for y

EX: $\frac{dy}{dx} = e^{y/x} + \frac{y}{x}$

Let $v = \frac{y}{x}$. Then $y = xv$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Plug into ODE: $v + x\frac{dv}{dx} = e^{v} + v$

Seperate: $\frac{dv}{e^v} = \frac{dx}{x}$

$$\int e^{-v}\,dv = \int \frac{dx}{x}$$

$$-e^{-v} = \ln|x| + C$$

Solve for v: $e^{-v} = -\ln|x| - C$

$$-v = \ln(-\ln|x| - C)$$

$$v = -\ln(\underbrace{-\ln|x| - C}_{\text{need this} > 0})$$

Last step: $v = \frac{y}{x}$

$$\boxed{y = -x\ln(-\ln|x| - C)}$$

p. 51: 36. $(x^2 + 3xy + y^2)\,dx + (-x^2)\,dy = 0$

Just another way to write ODE:

Divide by dx: $(-x^2)\frac{dy}{dx} = -(x^2+3xy+y^2)$

$$\frac{dy}{dx} = \frac{x^2+3xy+y^2}{x^2} = 1 + 3\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2$$

Homog!

EX: $\ln y - \ln x = \ln\frac{y}{x}$ Homog!

EX: $\frac{dy}{dx} = \frac{x+y}{x-y}\cdot\frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \frac{1+\frac{y}{x}}{1-\frac{y}{x}}$ Aha!

$$v + x\frac{dv}{dx} = \frac{1+v}{1-v}$$

$$x\frac{dv}{dx} = \frac{1+v}{1-v} - \underbrace{v}_{\frac{v(1-v)}{1-v}} = \frac{1+v-v(1-v)}{1-v} = \frac{1+v^2}{1-v}$$

$$\int\frac{1-v}{1+v^2}\,dv = \int\frac{dx}{x}$$

$$\underbrace{\int\frac{1}{1+v^2}\,dv}_{\tan^{-1} v} - \underbrace{\int\frac{v}{\underbrace{1+v^2}_{u}}\,dv}_{\int\frac{\frac{1}{2}du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln(1+v^2)} = \ln|x| + C$$

then $du = 2v\,dv$ and $v\,dv = \frac{1}{2}du$

$$\tan^{-1} v - \frac{1}{2}\ln(1+v^2) = \ln|x| + C$$

Last: $\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(1+\left(\frac{y}{x}\right)^2\right) = \ln|x| + C$ ← Find C, given $y(x_0)=y_0$

Gives y implicitly as a fcn of x.

Advice: Given a choice: Linear, separable, homogeneous, ...