Lesson 10 Exact eqns

Prob: Find an ODE solved by family of curves

$$x^2 y + y^3 = C \leftarrow \text{defines } y(x) \text{ implicitly}$$

Take $\frac{d}{dx}$ of eqn and use chain rule:

$$\frac{d}{dx}(x^2 y) + \frac{d}{dx}(y^3) = 0$$

$$\left[(2x)y + x^2 \frac{dy}{dx}\right] + 3y^2 \frac{dy}{dx} = 0$$

$$\boxed{(2xy) + (x^2 + 3y^2)\frac{dy}{dx} = 0}$$

Want to go other way.

Chain rule: $\frac{d}{dx}\varphi(u,v) = \frac{\partial \varphi}{\partial u}\frac{du}{dx} + \frac{\partial \varphi}{\partial v}\frac{dv}{dx}$

Suppose that $\varphi(x,y) = C \leftarrow$ defines family of curves $y = y(x)$.

$$\frac{d}{dx}\left[\varphi(x,y)\right] = \frac{d}{dx} C$$

$$\frac{\partial \varphi}{\partial x}\underbrace{\frac{dx}{dx}}_{1} + \frac{\partial \varphi}{\partial y}\frac{dy}{dx} = 0$$

$$\boxed{\frac{\partial \varphi}{\partial x}(x,y) + \frac{\partial \varphi}{\partial y}(x,y)\frac{dy}{dx} = 0}$$

Aha! Given an ODE $M(x,y) + N(x,y)\frac{dy}{dx} = 0$,

look for $\varphi(x,y)$ such that $\frac{\partial \varphi}{\partial x} = M$, $\frac{\partial \varphi}{\partial y} = N$.

If we get φ, then $\varphi(x,y) = C$ gives sol'n to ODE!

<u>Famous problem</u>: Given a force field $\vec{F} = M\hat{i} + N\hat{j}$, find a <u>potential</u> <u>fcn</u> φ such that $\nabla\varphi = \vec{F}$.

Condition needed: $\text{Curl } \vec{F} = \vec{0}$

$$\vec{F} = M\hat{i} + N\hat{j} + 0\hat{k}$$

$$\text{Curl } \vec{F} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{bmatrix} = \underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)}_{\text{Need } = 0} \hat{k}$$

<u>Big fact</u>: Can find φ such that $\frac{\partial \varphi}{\partial x} = M$ and $\frac{\partial \varphi}{\partial y} = N$ exactly when $\boxed{\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}}$.

Easy direction: If $\frac{\partial \varphi}{\partial x} = M$ and $\frac{\partial \varphi}{\partial y} = N$

$\frac{\partial}{\partial y}$: $\frac{\partial^2 \varphi}{\partial y \partial x} = \frac{\partial M}{\partial y}$ $\qquad$ $\frac{\partial}{\partial x}$: $\frac{\partial^2 \varphi}{\partial x \partial y} = \frac{\partial N}{\partial x}$

equal if φ is C^2-smooth.

<u>Notation and how to remember</u>: $M + N\frac{dy}{dx} = 0$ ← mult by dx

differential 1-form ⟶ $M\,dx + N\,dy = 0$

exat differential ⟶ $d\varphi = \frac{\partial \varphi}{\partial x}\,dx + \frac{\partial \varphi}{\partial y}\,dy$

Need $\frac{\partial}{\partial x}\Big[\underbrace{\text{Thing in front of } dy}_{N}\Big] = \frac{\partial}{\partial y}\Big[\underbrace{\text{Thing in front of } dx}_{M}\Big]$

<u>EX</u>: $\underbrace{(y^2 + \cos x)}_{M} + \underbrace{(2xy + \sin y)}_{N}\frac{dy}{dx} = 0$

$$\frac{\partial N}{\partial x} \stackrel{?}{=} \frac{\partial M}{\partial y}$$

$$2y + 0 \stackrel{?}{=} 2y + 0 \checkmark$$

Can get a φ!

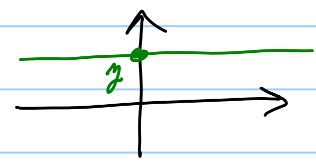
Want φ with $\begin{cases} \dfrac{\partial \varphi}{\partial x} = y^2 + \cos x & (A) \\[2mm] \dfrac{\partial \varphi}{\partial y} = 2xy + \sin y & (B) \end{cases}$

First use (A):

$$\varphi = \int y^2 + \cos x \, dx = xy^2 + \sin x + \underbrace{C(y)}_{\text{Arbitrary fcn of } y}$$

(Treat y like a const)

Why $C(y)$: If $\frac{\partial}{\partial x} f(x,y) \equiv 0$



$\leftarrow$ $f(x,y)$ is const along horiz lines

So $f(x,y) =$ fcn of y alone

From (A), got: $\boxed{\varphi = xy^2 + \sin x + C(y)}$

Next use (B):

$$\frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial y}\left[xy^2 + \sin x + C(y)\right] \stackrel{\text{want } = N}{=} 2xy + \sin y$$

$$2xy + 0 + C'(y) = 2xy + \sin y$$

$$\underline{C'(y) = \sin y}$$

So $C(y) = \int \sin y \, dy = -\cos y$

skip + const. here

Just want a φ that works.

Got $\underline{\varphi = xy^2 + \sin x - \cos y}$

Sol^n to ODE: $\boxed{xy^2 + \sin x - \cos y = K}$

EX: $\underbrace{\left(ye^{xy}+2x\right)}_{M} + \underbrace{\left(xe^{xy}-y\right)}_{N}\frac{dy}{dx} = 0$

Check: $\frac{\partial M}{\partial y} = 1\cdot e^{xy} + y\left[xe^{xy}\right] + 0$

$\frac{\partial N}{\partial x} = 1\cdot e^{xy} + x\left[ye^{xy}\right] - 0$

$= \checkmark$

ODE is "exact".

Want φ with $\begin{cases} \frac{\partial \varphi}{\partial x} = ye^{xy}+2x & (A) \\ \frac{\partial \varphi}{\partial y} = xe^{xy}-y & (B) \end{cases}$

Use (A): $\varphi = \int ye^{xy}+2x\,dx = \boxed{y\left[\frac{1}{y}e^{xy}\right]+x^2+C(y)}$

Use (B): $\frac{\partial}{\partial y}\left[e^{xy}+x^2+C(y)\right] \overset{\text{want}}{=} xe^{xy}-y$

$xe^{xy}+0+C'(y) \overset{\text{want}}{=} xe^{xy}-y$

$\boxed{C'(y) = -y}$

So $\underline{C(y) = \int -y\,dy = -\frac{1}{2}y^2}$

No $+C$.

Get $\boxed{\varphi = e^{xy}+x^2-\frac{1}{2}y^2 = K}$

What to do if $M\,dx + N\,dy = 0$ is not exact?

Hmmm. Multiply by $u(x,y)$ to make it exact.

$$(uM)\,dx + (uN)\,dy = 0$$

Need $\frac{\partial}{\partial y}\left[uM\right] = \frac{\partial}{\partial x}\left[uN\right]$ ← Ugh! PDE in u

Hmmm. What if u could be a fcn of x alone?

Need $uM_y = u'N + uN_x$

$$\boxed{\frac{u'}{u} = \frac{M_y - N_x}{N}}$$

← works if the RHS is a fcn of x alone.