Lesson 14 $\quad 2^{nd}$ order linear ODE (3.2)

EX: $L[y] = y'' + 3y' + 2y = 0$ ← Homogeneous

$$\begin{cases} y(0) = 7 \\ y'(0) = 8 \end{cases}$$

Last time: Tried $y = e^{rx}$. Got $L[e^{rx}] = \underbrace{(r^2+3r+2)}_{\text{need } = 0} e^{rx} = 0$ ← want

$r = -1, -2$. Got two sol^{n}s e^{-x}, e^{-2x}.

Supperposition principle: $c_1 e^{-x} + c_2 e^{-2x}$ is also a sol^n.

Sol^n: $\quad y = c_1 e^{-x} + c_2 e^{-2x}$ $\qquad$ want

$\quad y' = -c_1 e^{-x} - 2c_2 e^{-2x}$

$$\begin{cases} y(0) = c_1 + c_2 = 7 \\ y'(0) = -c_1 - 2c_2 = 8 \end{cases}$$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}}_{\det \neq 0\,!} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

Cramer's Rule:

$$c_1 = \frac{\det \begin{bmatrix} 7 & 1 \\ 8 & -2 \end{bmatrix}}{\det \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}} = \frac{-22}{-1} = \underline{\underline{22}}$$

$$c_2 = \frac{\det \begin{bmatrix} 1 & 7 \\ -1 & 8 \end{bmatrix}}{\det \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}} = \frac{15}{-1} = \underline{\underline{-15}}$$

Sol^n: $\quad y = 22e^{-x} - 15e^{-2x}$ is <u>the</u> sol^n with $\begin{cases} y(0) = 7 \\ y'(0) = 8 \end{cases}$

↑ U of E & U.

Wronskian: If y_1 and y_2 solve a 2^{nd} order linear homog eqn, then the <u>Wronskian</u> of y_1 and y_2 is

$$W = W[y_1, y_2] = W[y_1, y_2](x)$$

$$= \det \begin{bmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{bmatrix} = y_1 y_2' - y_2 y_1'$$

Hmmm: $-\dfrac{W[y_1, y_2]}{y_1^2} = \dfrac{d}{dx}\left(\dfrac{y_1}{y_2}\right)$

Why Wronskian? $y = c_1 y_1 + c_2 y_2$ solves ODE.

$y' = c_1 y_1' + c_2 y_2'$

IVP: $\begin{cases} y(x_0) = c_1 y_1(x_0) + c_2 y_2(x_0) = A \\ y'(x_0) = c_1 y_1'(x_0) + c_2 y_2'(x_0) = B \end{cases}$ want

$$\underbrace{\begin{bmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{bmatrix}}_{\det = W(x_0).\ \text{Need} \neq 0!} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

Fact: If you find two sol'^{n}s y_1, y_2 with $W[y_1, y_2](x_0) \neq 0$, then $y = c_1 y_1 + c_2 y_2$ is the gen^l sol^n (since we can solve any IVP's with it).

Fun fact: $y'' + P(x) y' + Q(x) y = 0$

Plug in y_1, y_2. Algebra. $\dfrac{W'}{W} = -P(x)$

Abel's formula $\longrightarrow W[y_1, y_2] = C e^{-\int P(x)\,dx} \longleftarrow$ Aha! $W \equiv 0$ or never zero

Consequently: Only need to check $W \neq 0$ at one point.

(assuming $P(x)$ is continuous.)

EX: $y'' + 4y' + 4y = 0$

Try $y = e^{rx}$. $\quad y' = re^{rx}$

$y'' = r^2 e^{rx}$

Want $(r^2 e^{rx}) + 4(re^{rx}) + 4(e^{rx})$

$$\underbrace{[r^2 + 4r + 4]}_{\text{"Characteristic poly"}} e^{rx} = 0$$

Need $r^2 + 4r + 4 = 0$

$(r+2)^2 = 0 \leftarrow$ Ouch! Only get $y_1 = e^{-2x}$

Hmmm: $\underbrace{\frac{\partial}{\partial r} L[e^{rx}]} = \frac{\partial}{\partial r}(r+2)^2 e^{rx} \leftarrow$ Euler: Diff w.r.t. r!

$L[\underbrace{\frac{\partial}{\partial r} e^{rx}}_{xe^{rx}}] = 2(r+2)e^{rx} + (r+2)^2 xe^{rx} \leftarrow$ Let $r=-2$!

get $L[xe^{-2x}] = 0$. Aha! $y_2 = xe^{-2x}$ is a second sol^n.

Big question: Is $c_1 e^{-2x} + c_2 xe^{-2x}$ the gen^l sol^n?

Check Wronskian: $W = \det \begin{bmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & (1 \cdot e^{-2x} - 2xe^{-2x}) \end{bmatrix}$

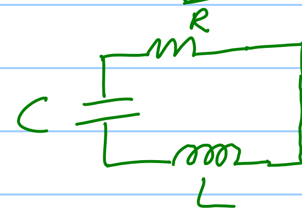
$$= \det \begin{bmatrix} e^{-2x} & 0 \\ -2e^{-2x} & e^{-2x} \end{bmatrix} = e^{-2x}e^{-2x} - 0 = e^{-4x}$$

$-x(\text{col } 1) + (\text{Col } 2)$

never zero

$= 1 \cdot e^{-\int P\,dx}$ ✓

So $y = c_1 e^{-2x} + c_2 x e^{-2x}$ is gen^l sol^n!



Big fact: $ay'' + by' + cy = 0$

Char eqn: $ar^2 + br + c = 0$

Get roots r_1, r_2.

Case 1: r_1, r_2 are real and unequal. $b^2 - 4ac > 0$

$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ is gen^l sol^n $\left(W = (r_2 - r_1) e^{(r_1 + r_2)x} \right.$

Case 2: $r_1 = r_2$ repeated root case. $b^2 - 4ac = 0$

$y = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$ is gen^l sol^n.

Case 3: r_1, r_2 complex $b^2 - 4ac < 0$

$r = A \pm Bi$

$y = c_1 e^{Ax} \cos Bx + c_2 e^{Ax} \sin Bx$ is gen^l sol^n.

Euler: $r = A + Bi$ Who cares? $e^{(A+Bi)x}$ is a "sol^n"

$$y = e^{(A+Bi)x} = e^{Ax + (Bx)i} = e^{Ax} e^{(Bx)i}$$
$$= e^{Ax} (\cos Bx + i \sin Bx)$$

Euler: $e^{i\theta} = \cos\theta + i\sin\theta$

$$y = \Big(\underbrace{e^{Ax}\cos Bx}_{\text{real part is a sol}^n}\Big) + i\Big(\underbrace{e^{Ax}\sin Bx}_{\text{imag part is a sol}^n}\Big) \leftarrow \text{complex sol}^n \text{ to ODE}$$

Wronskian $\neq 0$! Get <u>two</u> real sol^ns from <u>one</u> complex sol^n and $W \neq 0$.