Lesson 15 Complex roots (3.3)

Written 13,14,15 next Wed
WebAssign 15 due Mon.

Abel's formula: $W = y_1 y_2' - y_1' y_2$

$W' = y_1 y_2'' - y_1'' y_2$

$$\begin{cases} y_2'' + P y_2' + Q y_2 = 0 & \leftarrow \text{mult by } y_1 \\ y_1'' + P y_1' + Q y_1 = 0 & \leftarrow \text{mult by } y_2 \end{cases} \quad -$$

$$\underbrace{W' + P W}_{\text{Linear first order:}} + 0 = 0 \qquad W = C e^{-\int P dx} \leftarrow$$

$W \equiv 0$ or never zero! Only need to test W at one pt!

2nd order linear homogeneous ODE with const. coeff

$$a y'' + b y' + c y = 0 \qquad (\text{Try } y = e^{rx})$$

Characteristic Eqn: $a r^2 + b r + c = 0$.

Today: Complex roots case: $r = A \pm B i$

$b^2 - 4ac < 0$

$B = \dfrac{\sqrt{|b^2 - 4ac|}}{2a}$

$A = \dfrac{-b}{2a}$

Euler: $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$

$$e^{ix} = 1 + ix + i^2 \frac{x^2}{2!} + i^3 \frac{x^3}{3!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= \cos x + i \sin x \leftarrow \text{Euler's identity}$$

$$e^{a+bi} = e^a e^{bi} = e^a(\cos b + i \sin b) = (e^a \cos b) + i(e^a \sin b)$$

$$e^{\overbrace{(A+Bi)}^{r}x} = \underbrace{(e^{Ax} \cos Bx)}_{u(x)} + i \underbrace{(e^{Ax} \sin Bx)}_{v(x)}$$

$$\Psi(x) = u(x) + i\,v(x)$$
$$\Psi'(x) = u'(x) + i\,v'(x)$$

$$\frac{\Psi(x+h) - \Psi(x)}{h}$$

Let $h \to 0 \ldots$

$$L[\Psi] = a\Psi'' + b\Psi' + c\Psi = 0$$ Ψ complex sol^n

$$L[u+iv] = \underbrace{L[u]}_{u \text{ is a real } sol^n} + i\,\underbrace{L[v]}_{v \text{ is a real } sol^n} = 0$$

Get two real sol^ns from one complex sol^n.

Big question: $u = e^{Ax}\cos Bx \qquad v = e^{Ax}\sin bx$

Is $W[u,v] \neq 0$?

$$W = \det\begin{bmatrix} e^{Ax}\cos Bx & e^{Ax}\sin Bx \\ Ae^{Ax}\cos bx - Be^{Ax}\sin Bx & Ae^{Ax}\sin Bx + Be^{Ax}\cos Bx \end{bmatrix}$$

$$= \det\begin{bmatrix} e^{Ax}\cos Bx & e^{Ax}\sin Bx \\ -Be^{Ax}\sin Bx & Be^{Ax}\cos Bx \end{bmatrix}$$

$$= Be^{Ax}e^{Ax}\underbrace{\det\begin{bmatrix} \cos & \sin \\ -\sin & \cos \end{bmatrix}}_{1} = Be^{2Ax} \leftarrow \text{not zero!}$$

Fantastic! $y = c_1 e^{Ax}\cos Bx + c_2 e^{Ax}\sin Bx$ is gen^l sol^n!

Why it works: $\frac{d}{dx}\left[e^{(A+Bi)x}\right] = (A+Bi)\,e^{(A+Bi)x}$ $\leftarrow$ routine to check.

$$\frac{d}{dx}\left[e^{rx}\right] = re^{rx}$$ even when r is complex.

$L[e^{rx}] = \underbrace{(ar^2 + br + c)}_{=0 \text{ if } r \text{ is complex root.}} e^{rx}$ even if r is complex

Get complex sol^n!

Note: We only used one of them: $e^{(A+Bi)x}$.

The other, $e^{(A-Bi)x} = e^{Ax} \underbrace{\cos(-Bx)}_{=\cos Bx} + i e^{Ax} \underbrace{\sin(-Bx)}_{-\sin Bx}$

gives same real gen^l sol^n.

In EE $y = c_1 e^{(A+Bj)x} + c_2^{(A-Bj)x}$

EX: $y'' + 4y' + 13y = 0$ $\begin{cases} y(0)=5 \\ y'(0)=7 \end{cases}$

$m=1$ $c=4$ $k=13$

$L=1$ $R=4$ $\frac{1}{C}=13$

$r^2 + 4r + 13 = 0$ $r =$ quad formula

$(r + \overset{\frac{1}{2}\cdot 4}{2})^2 + 9 = 0$

$(r+2)^2 = -9$

$r+2 = \pm 3i$

$r = \underline{\underline{-2}} \pm 3i$

Complex sol^n: $e^{(-2+3i)x}$

$e^{-2x} e^{3ix} = (\underbrace{e^{-2x}\cos 3x}_{u}) + i(\underbrace{e^{-2x}\sin 3x}_{v})$

Get real gen^l sol^n: $y = \underline{c_1 e^{-2x}\cos 3x + c_2 e^{-2x}\sin 3x}$

$y' = c_1(-2e^{-2x}\cos 3x - 3e^{-2x}\sin x)$
$+ c_2(-2e^{-2x}\sin 3x + 3e^{-2x}\cos 3x)$

$\begin{cases} y(0) = c_1 \overset{\text{want}}{=} 5 \\ y'(0) = c_1(-2) + c_2(3) \overset{\text{want}}{=} 7 \end{cases}$

$$\begin{cases} c_1 = 5 \\ c_2 = 17/3 \end{cases}$$

$$3c_2 = 7 + 2\cdot 5 = 17$$

$$y(x) = 5e^{-2x}\cos 3x + \frac{17}{3}e^{-2x}\sin 3x$$