Lesson 21 Forced vibrations (3.8)

Written: Lessons 20, 21 due Wed
WebAssign: Lesson 21 due Mon.
Project 1: due Tues., March 20, 11 pm, uploaded to Blackboard

$$my'' + cy' + ky = F_0 \cos\omega t \leftarrow \text{"input"}$$

$$y = \underbrace{\begin{cases} c_1 e^{-\lambda_1 t} + c_2 e^{-\lambda_2 t} \\ c_1 e^{-\lambda t} + c_2 t e^{-\lambda t} \\ A e^{-\frac{c}{2m}t}\cos(\mu t - \varphi) \end{cases}}_{\text{homog sol}^n} + \underbrace{\frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2 + c^2\omega^2}}\cos(\omega t - \eta)}_{y_p \text{ part. sol}^n}$$

homog sol^n — Transient sol^n

y_p part. sol^n — Steady State "output" or Response

Where $\omega_0 = \sqrt{\frac{k}{m}}$, $\mu = \frac{\sqrt{|c^2 - 4mk|}}{2m}$

Notation $\Delta = \sqrt{m^2(\omega_0^2-\omega^2)^2 + c^2\omega^2}$, $\eta = \sin^{-1}\frac{c\omega}{\Delta}$

Prob: What ω gives biggest response?

$\frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2 + c^2\omega^2}}$ is max when $\sqrt{}$ is min.

and $\sqrt{}$ is min when $m^2(\omega_0^2-\omega^2)^2 + c^2\omega^2$ is min.

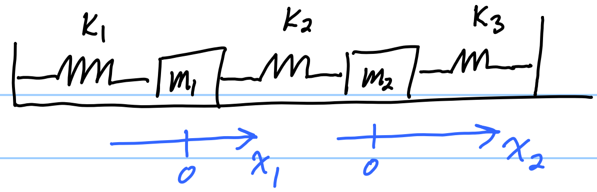
Freshman calc! Set slope $= 0$. Get

$$\omega_{max} = \omega_0\sqrt{1 - \frac{c^2}{2mk}} \leftarrow \text{if } 1 - \frac{c^2}{4mk} < 0, \text{ the max happens at } \omega = 0.$$

Read end of 3.8. Interesting: Dimensionless constants.

Higher order linear ODEs:

Prob:



0's at slack position.

No friction, massless springs

$F = ma$ for m_1: $(\text{Net force}) = m_1 a$

$$\boxed{-k_1 x_1 + k_2 (x_2 - x_1) = m_1 \ddot{x}_1}$$

for m_2: $-k_3 x_2 + \underbrace{\left[-k_2 (x_2 - x_1)\right]}_{\text{equal and opposite}} = m_2 \ddot{x}_2$

$$\boxed{\begin{aligned} (-k_1 - k_2) x_1 + k_2 x_2 &= m_1 \ddot{x}_1 \\ k_2 x_1 + (-k_2 - k_3) x_2 &= m_2 \ddot{x}_2 \end{aligned}}$$

Take $k_1 = k_2 = k_3 = 1$, $m_1 = m_2 = 1$

$$\boxed{\begin{aligned} -2 x_1 + x_2 &= \ddot{x}_1 \\ x_1 - 2 x_2 &= \ddot{x}_2 \end{aligned}}$$

$x_2 = \ddot{x}_1 + 2 x_1$

So $\ddot{x}_2 = x_1^{(4)} + 2 \ddot{x}_1$

2nd eqn becomes: $x_1 - 2(\ddot{x}_1 + 2x_1) = \left[x_1^{(4)} + 2\ddot{x}_1\right]$

$$\boxed{x_1^{(4)} + 4\ddot{x}_1 + 3x_1 = 0}$$

Try $x_1 = e^{rt}$: $(r^4 e^{rt}) + 4(r^2 e^{rt}) + 3e^{rt} = 0$ want

$$\underbrace{\left[r^4 + 4r^2 + 3\right]}_{\text{must }=0} \underset{\uparrow \text{ never zero!}}{e^{rt}} = 0$$

$$(r^2+1)(r^2+3) = 0$$

$$r = \pm i, \ \pm\sqrt{3}\, i$$

$$x_1 = c_1 e^{it} + c_2 e^{-it} + c_3 e^{\sqrt{3}it} + c_4 e^{-\sqrt{3}it}$$

is cool, but ...

Big fact: One complex sol^n yield two real sol^{n}s.

$$e^{it} = \cos t + i \sin t \quad \begin{cases} \cos t \\ \sin t \end{cases}$$

$$e^{i\sqrt{3}t} = \cos\sqrt{3}t + i \sin\sqrt{3}t \quad \begin{cases} \cos\sqrt{3}t \\ \sin\sqrt{3}t \end{cases}$$

Four different real sol^{n}s!

Big fact 2: Supperposition princ holds! [Linear, homogeneous]

$$\boxed{x_1 = c_1 \cos t + c_2 \sin t + c_3 \cos\sqrt{3}t + c_4 \sin\sqrt{3}t}$$

Recall $\quad x_2 = \ddot{x}_1 + 2x_1$

$$= \left[-c_1\cos t - c_2 \sin t - 3c_3 \cos\sqrt{3}t - 3c_4 \sin\sqrt{3}t\right]$$

$$+ 2c_1 \cos t + 2c_2 \sin t + 2c_3 \cos\sqrt{3}t + 2c_4 \sin\sqrt{3}t$$

$$\boxed{x_2 = c_1 \cos t + c_2 \sin t - c_3 \cos\sqrt{3}t - c_4 \sin\sqrt{3}t}$$

Big question: 4 Initial Conditions:

$$x_1(0) = A_1 \qquad x_2(0) = A_2$$

$$\dot{x}_1(0) = B_1 \qquad \dot{x}_2(0) = B_2$$

Hmmm. 4 constants! Can I find x_1, x_2 satisfying any initial cond?

Yes! Wronskian = det $\begin{bmatrix} \cos t & \sin t & \cos\sqrt{3}t & \sin\sqrt{3}t \\ & \text{derivatives} & & \\ & \text{2nd deriv.} & & \\ & \text{3rd deriv.} & & \end{bmatrix}$

Not zero!

What are the basic sol^{n}s:

$c_1 = 1$, rest $= 0$ $\quad \begin{cases} x_1 = \cos t \\ x_2 = \cos t \end{cases}$ $\leftarrow$ Mode of oscillation

$c_3 = 1$, rest $= 0$ $\quad \begin{cases} x_1 = \cos\sqrt{3}t \\ x_2 = -\cos\sqrt{3}t \end{cases}$

IVP: $\quad x_1(0) = 0 \qquad x_2(0) = 1$

$\quad \dot{x}_1(0) = 0 \qquad \dot{x}_2(0) = 0$

Get $\begin{cases} x_1 = \frac{1}{2}\cos t - \frac{1}{2}\cos\sqrt{3}t = \sin\left(\frac{\sqrt{3}-1}{2}t\right)\sin\left(\frac{\sqrt{3}+1}{2}\right)t \\ x_2 = \frac{1}{2}\cos t + \frac{1}{2}\cos\sqrt{3}t = \cos\left(\frac{\sqrt{3}-1}{2}t\right)\cos\left(\frac{\sqrt{3}+1}{2}\right)t \end{cases}$