Lesson 23 Higher order linear ODE with constant coefficients

WebAssign: Lessons 22, 23 due Thurs

Last time: $y''' + y' = e^{-2t}$ $(\dagger)$

Got homog sol^n to $y''' + y' = 0$ $(*)$: $y = c_1 + c_2 \cos t + c_3 \sin t$

Need one particular sol^n y_P to $(\dagger)$. Try $y_P = Ae^{-2t}$.

$y' = -2Ae^{-2t}$ ... $y''' = (-2)^3 A e^{-2t} = -8A e^{-2t}$.

To make it work: $y''' + y' \overset{\text{want}}{=} e^{-2t}$

$$-8Ae^{-2t} + -2Ae^{-2t} = e^{-2t}$$

$$\underbrace{[-10A]}_{=1} e^{-2t} = e^{-2t} \qquad \boxed{A = \frac{-1}{10}}$$

Gen^l Sol^n to $(\dagger)$: $y = (c_1 + c_2 \cos t + c_3 \sin t) - \frac{1}{10} e^{-2t}$

To pin down c's, need $\underline{\underline{3}}$ initial conditions.

EX: $y(0) = 1$, $y'(0) = 2$, $y''(0) = 3$

$$y = c_1 + c_2 \cos t + c_3 \sin t - \tfrac{1}{10} e^{-2t}$$
$$y' = -c_2 \sin t + c_3 \cos t + \tfrac{1}{5} e^{-2t}$$
$$y'' = -c_2 \cos t - c_3 \sin t - \tfrac{2}{5} e^{-2t}$$

$$\begin{cases} y(0) = c_1 + c_2 - \frac{1}{10} \overset{\text{want}}{=} 1 & \xleftarrow{3} \text{ get } c_1 \\ y'(0) = \quad + c_3 + \frac{1}{5} = 2 & \xleftarrow{2} \text{ get } c_3 \\ y''(0) = \quad - c_2 - \frac{2}{5} = 3 & \xleftarrow{1} \text{ get } c_2 \end{cases}$$

Big fact: Polynomials with real coefficients factor into terms like this:

1) $(r - r_1)$, r_1 real — real root of multiplicity 1

2) $(r - r_1)^m$, r_1 real — real root of multiplicity m

3) (r^2+br+c) roots $r = A \pm Bi$
Complex roots of mult. 1

4) $(r^2+br+c)^m$ comp roots of mult m.

Hard problem: Factoring polynomials of high degree.

Rules for finding general solution of an n-th order linear homogeneous ODE with constant coeff

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

Step 1: Try $y = e^{rt}$: Plug in. get

$$\underbrace{P(r)}_{\text{need } =0} e^{rt} \overset{\text{want}}{=} 0$$

$$P(r) = a_n r^n + \cdots + a_1 r + a_0$$

Step 2: Factor $P(r)$ into basic terms:
Real roots r_j : $(r-r_j)^{m_j}$
Complex roots always appear in pairs: $A \pm Bi$
correspond to $(r^2+br+c)^m$

Ex: Get $P(r) = a_n (r-r_1)(r-r_2)^3(r^2+1)(r^2+2r+2)^5$

Rule: 1) Real root r_1 of mult 1:
Get one sol^n $y = e^{r_1 t}$

2) Real root r_1 of mult m_1:
$L[e^{rt}] = P(r)e^{rt}$ ← take $\frac{\partial}{\partial r}$ and set $r = r_1$

$$\frac{\partial}{\partial r} L\left[e^{rt}\right] = L\left[\underbrace{\frac{\partial}{\partial r} e^{rt}}_{te^{rt}}\right] = \underbrace{\frac{\partial}{\partial r}\left\{\left[(r-r_1)^{m_1} Q(r)\right] e^{rt}\right\}}_{\text{get zero on right}} \leftarrow \text{set } r=r_1$$

Works $m_1 - 1$ times!

Get m_1 sol^{n}s :

$$\left.\begin{array}{c} e^{r_1 t} \\ t\, e^{r_1 t} \\ t^2 e^{r_1 t} \\ \vdots \\ t^{m_1-1} e^{r_1 t} \end{array}\right\} \quad m_1 \text{ sol}^n\text{s!}$$

3) Case $\quad (r^2 + br + c)^1 \quad$ Complex roots $A \pm Bi$

Complex sol^n $\quad y = e^{(A+Bi)t} = \underbrace{e^{At}\cos Bt}_{} + i\, \underbrace{e^{At}\sin Bt}_{}$

two real sol^{n}s!

4) $\quad (r^2 + br + c)^m \quad$ Comp. roots $A \pm Bi$

$$\left.\begin{array}{lcl} e^{(A+Bi)t} & < & \text{two real sol}^n\text{s} \\ t\,e^{(A+Bi)t} & < & \begin{array}{l} t\,e^{At}\cos Bt \\ t\,e^{At}\sin Bt \end{array} \\ \vdots & & \\ t^{m-1} e^{(A+Bi)t} & < & \begin{array}{l} t^{m-1} e^{At}\cos Bt \\ t^{m-1} e^{At}\sin Bt \end{array} \end{array}\right\} \quad \text{Get } 2m \ \underline{\text{different}} \text{ sol}^n\text{s}$$

<u>Good news</u>: From a poly $P(r)$ of degree n, get n different sol^{n}s this way <u>and</u> the Wronskian is guaranteed to be non-zero!

EX: $y^{(4)} + 2y'' + y = 0$

$r^4 + 2r^2 + 1 = 0$

$(r^2+1)^2 = 0 \qquad r = \pm i$, each of mult 2.

$$y = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t$$

Method of undetermined coeff:

$$\underbrace{a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y}_{\text{Const coeff linear}} = \underbrace{F(x)}_{\text{special form}}$$

Try y_p:

$F(x)$	Try $y_p =$
e^{at}	$A e^{at}$
t^n or any poly of degree n	$A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0$
$\cos \omega t$, $\sin \omega t$	$A \cos \omega t + B \sin \omega t$
$e^{at} \cos \omega t$ or $e^{at} \sin \omega t$	$A e^{at} \cos \omega t + B e^{at} \sin \omega t$

$t^n e^{at}\cos\omega t$ or $t^n e^{at}\sin\omega t$	$(A_n t^n + \cdots + A_0)e^{at}\cos\omega t + (B_n t^n + \cdots + B_0)e^{at}\sin\omega t$

Danger: Method works great when no single piece of the trial sol^n solves the homog equ, but when a piece does ...

Rule: Get y_p from the table. If any single piece of the trial sol^n solves homog, replace trial sol^n by t^m (y_p from table) where m is smallest positive integer that clears the danger.

EX: $\underbrace{y^{(4)} + 2y'' + y} = t^2 \sin t$

homog sol^n: $c_1 \cos t + c_2 \sin t + c_3 t\cos t + c_4 t \sin t$

Table: $y_p = (A_2 t^2 + A_1 t + A_0)\cos t + (B_2 t^2 + B_1 t + B_0)\sin t$

One piece: $A_1 t \cos t$ ← ouch!
$B_0 \sin t$

Correct $y_p = t^2\Big((A_2 t^2 + A_1 t + A_0)\cos t + (B_2 t^2 + B_1 t + B_0)\sin t\Big)$

One piece: $A_0 t^2 \cos t$ ← safe!