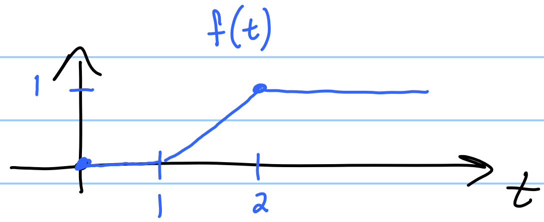
Lesson 26 Piecewise defined functions, s-shift, t-shift rules

Project 2 due Friday, 11 pm

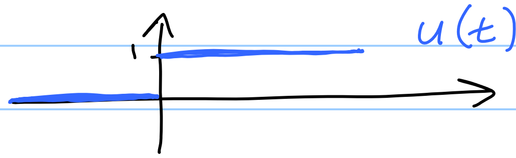
EX:



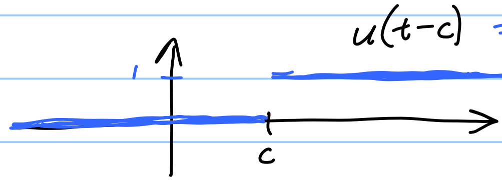
$$f(t) = \begin{cases} 0 & 0 \le t \le 1 \\ t-1 & 1 \le t \le 2 \\ 1 & t \ge 2 \end{cases}$$

Today: $\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt = 0 + \int_1^2 (t-1)e^{-st}\,dt + \int_2^\infty 1\cdot e^{-st}\,dt$

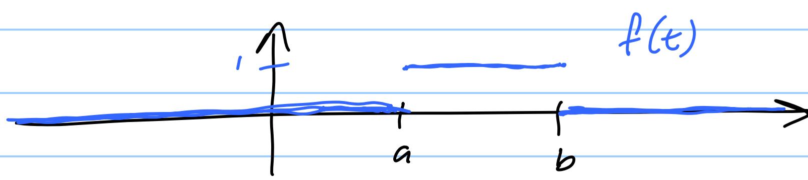
Def^n: The "Heaviside" function, or unit step function $u(t)$ or $u_0(t)$



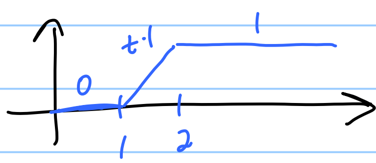
$u(t-c) = u_c(t)$



Turn on at $t=a$ and off at $t=b$:



$f(t) = u_a(t) - u_b(t)$



$$f(t) = 0\cdot\left[u_0(t) - u_1(t)\right] + (t-1)\left[u_1(t) - u_2(t)\right] + 1\cdot\left[u_2(t)\right]$$

Two big facts: 1) s-shifting rule: $\mathcal{L}[f(t)] = F(s)$

$\mathcal{L}\left[e^{at} f(t)\right] = F(s-a) \leftarrow$ shifting by a

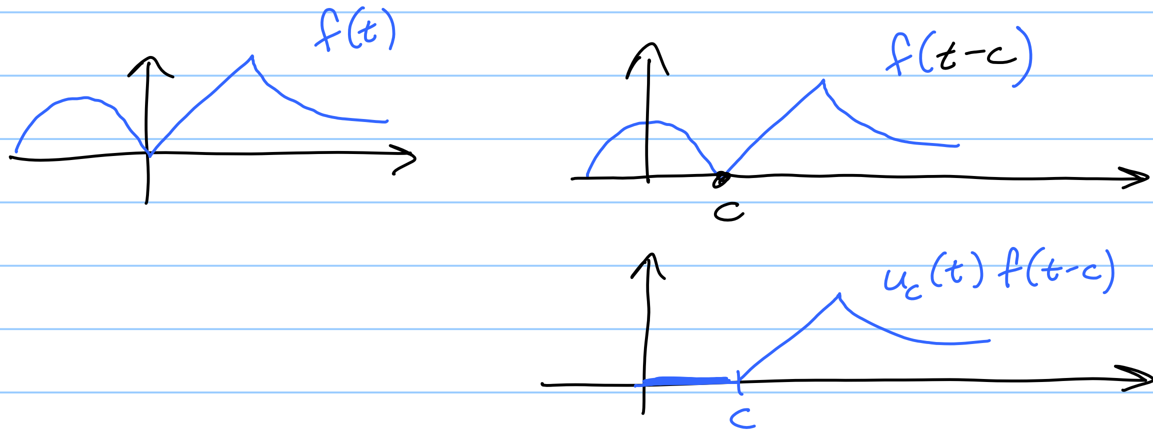
2) t-shifting rule: $\mathcal{L}\left[u_c(t) f(t-c)\right] = e^{-cs} F(s)$

Take $f(t) \equiv 1$ in #2 : $\mathcal{L}[1] = \frac{1}{s}$. So $\mathcal{L}[u_c(t)] = \dfrac{e^{-cs}}{s}$

Why 1 : $\mathcal{L}[e^{at} f(t)] = \int_0^\infty e^{at} f(t) e^{-st}\, dt = \int_0^\infty e^{-(s-a)t} f(t)\, dt$

$= F(s-a)$ ✓

Why 2 :



$$\mathcal{L}[u_c(t) f(t-c)] = \int_0^\infty u_c(t) f(t-c) e^{-st}\, dt$$

$$= \int_c^\infty f(\underbrace{t-c}_{\tilde{\tau}}) e^{-st}\, dt$$

$\tilde{\tau} = t - c$ $\quad$ $t = \tilde{\tau} + c$

$d\tilde{\tau} = dt$

When $t = c$: $\tilde{\tau} = 0$

as $t \to \infty$, so does $\tilde{\tau}$

$$= \int_{\tilde{\tau}=0}^\infty f(\tilde{\tau}) \underbrace{e^{-s(\tilde{\tau}+c)}}_{e^{-s\tilde{\tau}} e^{-sc}} d\tilde{\tau}$$

$$= e^{-sc} \int_{\tilde{\tau}=0}^\infty f(\tilde{\tau}) e^{-s\tilde{\tau}}\, d\tilde{\tau} = e^{-sc} F(s) \quad ✓$$

Important formulas : $\mathcal{L}[\sin bt] = \dfrac{b}{s^2+b^2}$

$\mathcal{L}[\cos bt] = \dfrac{s}{s^2+b^2}$

s-shift rule : $\mathcal{L}[e^{at} \sin bt] = \dfrac{b}{(s-a)^2+b^2}$

$\mathcal{L}[e^{at} \cos bt] = \dfrac{(s-a)}{(s-a)^2+b^2}$

EX: $\dfrac{3s+4}{\underbrace{s^2+2s+2}} = \dfrac{3s+4}{(s+1)^2+1} = \dfrac{3[(s+1)-1]+4}{(s+1)^2+1}$

$= 3\cdot\dfrac{(s+1)}{(s+1)^2+1^2} + \dfrac{1}{(s+1)^2+1^2}$ $\qquad a=-1$, $b=1$

$= \mathcal{L}\Big[\underbrace{3e^{-t}\cos t + e^{-t}\sin t}_{\mathcal{L}^{-1}[\ldots]}\Big]$

EX: Find $\mathcal{L}[\underbrace{u(t-3)}_{u_3(t)}\,t^2]$ $\qquad \mathcal{L}[u_c(t)f(t-c)] = e^{-cs}F(s)$

Big trick: $s = [(s-3)+3]$

$u_3(t)\,t^2 = u_3(t)\left[(t-3)+3\right]^2 = u_3(t)\Big[\underbrace{(t-3)^2+6(t-3)+9}\Big]$

Aha! $= f(t-3)$ where

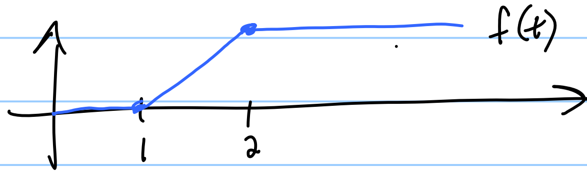
$f(\ \ ) = (\ \ )^2 + 6(\ \ ) + 9$

$f(t) = t^2 + 6t + 9$

So $F(s) = \dfrac{2}{s^3} + \dfrac{6}{s^2} + \dfrac{9}{s}$

$\mathcal{L}[u_3(t)t^2] = e^{-3s}F(s) = e^{-3s}\left[\dfrac{2}{s^3} + \dfrac{6}{s^2} + \dfrac{9}{s}\right]$

EX:



$f(t) = (t-1)[u_1 - u_2] + (1)\,u_2$

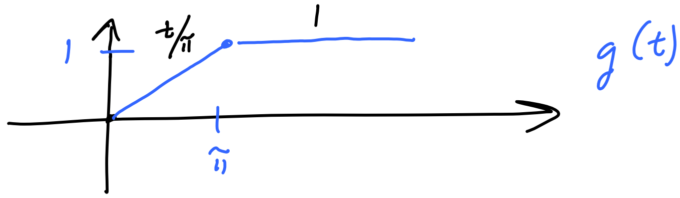
$= (t-1)u_1 + (-t+2)u_2$

$\mathcal{L}[u_c(t)f(t-c)] = e^{-cs}F(s)$

$\mathcal{L}\Big[u_1(t)\underbrace{(t-1)}_{f(t-1)=t-1} - \underbrace{(t-2)}_{g(t-2)=t-2}u_2(t)\Big]$

$f(t) = t$ $\qquad g(t) = t$

$= e^{-s}\cdot\dfrac{1}{s^2} - e^{-2s}\cdot\dfrac{1}{s^2}$

Prob: $y'' + y =$ 

$y(0) = 0$
$y'(0) = 0$

$$\left(s^2 Y - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0}\right) + Y = G(s)$$

$$g(t) = \left[u_0(t) - u_\pi(t)\right]\frac{t}{\pi} + \left[u_\pi(t)\right]\cdot 1$$

$$= \underbrace{u_0(t)\frac{t}{\pi}} - u_\pi(t)\frac{t}{\pi} + \underbrace{u_\pi(t)}$$

$\mathcal{L}[u_\pi] = e^{-\pi s}/s$

same as $u_0(t)\frac{t-0}{\pi}$

same as $\frac{t}{\pi}$ from point of view of $\mathcal{L}$ because $\int_0^\infty$

$$\mathcal{L}\left[u_0(t)\frac{t-0}{\pi}\right] = e^{-0\cdot s}\,\mathcal{L}\left[\frac{t}{\pi}\right] = \frac{1}{\pi}\cdot\frac{1}{s^2} = \mathcal{L}\left[\frac{t}{\pi}\right] \checkmark$$

Middle one: $\mathcal{L}\left[u_\pi(t)\, f(t-\pi)\right] = e^{-\pi s} F(s)$

$$\mathcal{L}\left[u_\pi(t)\frac{1}{\pi}\underbrace{\left[(t-\pi) + \pi\right]}_{=t}\right] = \frac{1}{\pi} e^{-\pi s}\,\mathcal{L}\left[t+\pi\right]$$

$$= \frac{1}{\pi} e^{-\pi s}\left[\frac{1}{s^2} + \frac{\pi}{s}\right]$$

$f(t-\pi) = (t-\pi) + \pi$

$f(\ \ ) = (\ \ ) + \pi$

$f(t) = t + \pi$

$$G(s) = \frac{1}{\pi}\cdot\frac{1}{s^2} - \frac{1}{\pi}e^{-\pi s}\left[\frac{1}{s^2} + \frac{\pi}{s}\right] + e^{-\pi s}\cdot\frac{1}{s}$$