Lesson 28 Impulse (Dirac Delta) function

$\mathcal{L}[f] = F$

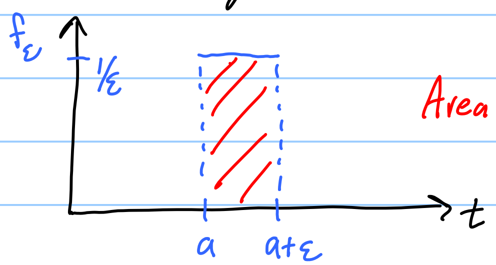
s-shift rule: $\mathcal{L}[e^{at} f(t)] = F(s-a)$

t-shift rule: $\mathcal{L}[u_c(t) f(t-c)] = e^{-cs} F(s)$

Amplitude of $c_1 \cos \omega t + c_2 \sin \omega t = A \cos(\omega t - \varphi)$

where $A = \sqrt{c_1^2 + c_2^2}$

Good sledge hammer:



Area = 1 for unit impulse = $\int$ force dt

$$f_\varepsilon(t) = \frac{1}{\varepsilon}\left[u_a(t) - u_{a+\varepsilon}(t)\right]$$

$$F_\varepsilon(s) = \frac{1}{\varepsilon}\left(\frac{e^{-as}}{s} - \frac{e^{-(a+\varepsilon)s}}{s}\right) = -\frac{1}{s}\Bigg[\underbrace{\frac{e^{-(a+\varepsilon)s} - e^{-as}}{\varepsilon}}_{\text{DQ!}}\Bigg]$$

Hmmm. As $\varepsilon \to 0$, f_ε flakes out, but

$$F_\varepsilon(s) \to -\frac{1}{s}\left[-se^{-as}\right] = e^{-as}$$

$g(\tau) = e^{-(\tau)s}$

$DQ = \dfrac{g(a+\varepsilon) - g(a)}{\varepsilon} \to g'(a)$

$g'(a) = -se^{-as}$

Think $\delta_a(t) \text{ "=" } \lim_{\varepsilon \to 0} f_\varepsilon(t)$

Formula: $\mathcal{L}[\delta_a(t)] = e^{-as}$

$\mathcal{L}[\delta_0(t)] = 1$

Problem: $y'' + y = f_\varepsilon(t)$ $\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$

Hit with $\mathcal{L}$: $\left[s^2 Y - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0}\right] + Y = F_\varepsilon(s)$

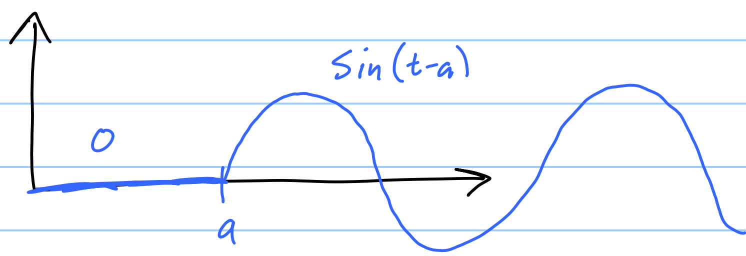
$$(s^2+1)Y = F_\varepsilon(s)$$

$$Y = \underbrace{\frac{1}{s^2+1}}_{\text{Transfer fcn}} F_\varepsilon(s)$$

Big idea. Let $\varepsilon \to 0$.

$$Y \xrightarrow[\varepsilon \to 0]{} \frac{1}{s^2+1} e^{-as}$$

Then $$y = \mathcal{L}^{-1}\left[e^{-as} \frac{1}{s^2+1} \right] = u_a(t) \sin(t-a)$$



Aha! Limit makes good sense on $\mathcal{L}$-side and can be used.

Perfect sledge hammer sol^n!

Physics: Impulse = Δ(momentum)

$F = ma$

$$\text{Impulse} = \int F\,dt = m \int a\,dt = m\,\Delta v$$

Key properties of $\delta_a(t)$:

1) $\delta_a(t) = 0$ if $t \neq a$

2) $\int \delta_a(t)\,dt = 1$

3) If h is continuous, then

$$\int h(t)\,\delta_a(t)\,dt = h(a)$$

Think limit f_ε as $\varepsilon \to 0$

$\delta_a(t)$ is not a true function. It is a distribution.

Consequence of 3: $$\int_0^\infty e^{-st}\,\delta_a(t)\,dt = e^{-sa} \quad \checkmark$$

EX: $y'' + 4y' + 13y = \delta_{\pi}(t)$ $\quad \begin{cases} y(0)=0 \\ y'(0)=0 \end{cases}$

$$\left[s^2 Y - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0}\right] + 4\left[sY - \underbrace{y(0)}_{0}\right] + 13\,Y = e^{-\pi s}$$

$$Y = \frac{1}{\underbrace{s^2+4s+13}_{H(s)}}\cdot e^{-\pi s} = H(s)e^{-\pi s}$$

So $\quad y = u_\pi(t)\, h(t-\pi)$

$$\frac{1}{s^2+4s+13} = \frac{1}{(s+2)^2+9} = \frac{1}{3}\cdot\frac{3}{(s+2)^2+3^2}$$

$\frac{1}{2}\cdot 4$

Table: $\quad \mathcal{L}[\sin bt] = \dfrac{b}{s^2+b^2} \quad \leftarrow b=3$

s-shift rule: $\quad \mathcal{L}[e^{at}f(t)] = F(s-a) \quad \leftarrow a=-2$

$$\mathcal{L}\left[e^{-2t}\sin 3t\right] = \frac{3}{(s+2)^2+3^2}$$

So $\quad h(t) = \frac{1}{3}e^{-2t}\sin 3t \quad$ and

$$y = u_\pi(t)h(t-\pi) = u_\pi(t)\frac{1}{3}e^{-2(t-\pi)}\sin 3(t-\pi)$$

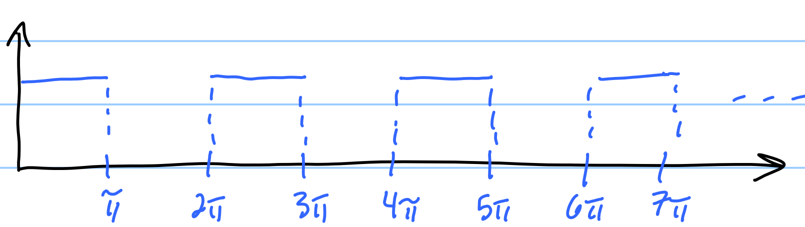
$$= \frac{1}{3}u_\pi(t)e^{-2t}e^{2\pi}\underbrace{\sin(3t-3\pi)}_{-\sin 3t}$$

$$= -\frac{e^{2\pi}}{3}u_\pi(t)e^{-2t}\sin 3t$$

$$= \begin{cases} 0 & 0\le t\le \pi \\ -\dfrac{e^{2\pi}}{3}e^{-2t}\sin 3t & t\ge \pi \end{cases}$$

Prob:

$$y'' + y = g(t) =$$



$$\begin{cases} y(0)=0 \\ y'(0)=0 \end{cases}$$

$$= \left[u_0(t) - u_\pi(t)\right] + \left[u_{2\pi}(t) - u_{3\pi}(t)\right] + \cdots$$

$$(s^2+1)Y = G(s) = \frac{1}{s}\left(e^{-0\cdot s} - e^{-\pi s}\right) + \frac{1}{s}\left(e^{-2\pi s} - e^{-3\pi s}\right) + \cdots$$

$$= \frac{1}{s}\left(1 - e^{-\pi s} + e^{2\pi s} - \underset{\uparrow\ (e^{-\pi s})^3}{e^{-3\pi s}} + \cdots\right)$$

$1 + r + r^2 + \cdots = \frac{1}{1-r}$ if $|r| < 1$

Aha! $r = -e^{-\pi s}$

$$G(s) = \frac{1}{s} \cdot \frac{1}{1 + e^{-\pi s}} \leftarrow \text{Cool!}$$

$$Y = \frac{1}{s(s^2+1)}\left(1 - e^{-\pi s} + e^{2\pi s} - \cdots\right)$$

$H(s) = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$ by partial fractions.

So $h(t) = 1 - \cos t$

$$y = h(t) - u_\pi(t)h(t-\pi) + u_{2\pi}(t)h(t-2\pi) - \cdots$$

Easy Fact: $\cos(t - (\text{Even})\pi) = \cos t$

$\cos(t - (\text{ODD})\pi) = -\cos t$

Get $y = \left(u_0 - u_\pi + u_{2\pi} - \cdots\right) \leftarrow g(t)$

$$- \cos t \left(\underbrace{1 + u_\pi + u_{2\pi} + u_{3\pi} + \cdots}_{\text{Staircase fcn! Resonance!}}\right)$$