# Lesson 31 Systems

2nd order ODE $\longleftrightarrow$ 2x2 system

EX: $\underset{\substack{\uparrow\\ m=1}}{u''} + \underset{\substack{\uparrow\\ c=5}}{5u'} + \underbrace{(1+\varepsilon u^2)}_{k}u = \cos\omega t$

ode45 demands that you convert to a 2x2 system

Step 1: Let $x_1(t) = u(t)$

Step 2: Let $x_2(t) = u'(t)$

System: $\frac{dx_1}{dt} = \sim\!\sim \longleftarrow$ easy!

$$\boxed{\frac{dx_1}{dt} = u' = x_2}$$

$\frac{dx_2}{dt} = \sim\!\sim$

$$\frac{dx_2}{dt} = u'' = \left[-5u' - (1+\varepsilon u^2)u + \cos\omega t\right] \longleftarrow \text{from ODE}$$

$$= -5x_2 - (1+\varepsilon x_1^2)x_1 + \cos\omega t$$

Get 2x2 system:
$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\[2ex] \dfrac{dx_2}{dt} = -5x_2 - x_1 - \varepsilon x_1^3 + \cos\omega t \end{cases}$$

Important problem:
$$\begin{cases} \dfrac{dx_1}{dt} = 5x_1 - x_2 \\[2ex] \dfrac{dx_2}{dt} = 3x_1 + x_2 \end{cases}$$

2x2 linear homog first order system with const coeff

Fancy notation: $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. $\quad \vec{x}\,' = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$

$$\vec{x}\,' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}\vec{x} = A\vec{x}$$

High school method gave $\vec{x} = c_1\begin{pmatrix} A \\ B \end{pmatrix}e^{r_1 t} + c_2\begin{pmatrix} C \\ D \end{pmatrix}e^{r_2 t}$

Big idea: Try sol$^n$'s of form $\vec{x} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{rt} = \vec{a}e^{rt}$.
$$\begin{cases} x_1 = a_1 e^{rt} \\ x_2 = a_2 e^{rt} \end{cases}$$

Want

$$\begin{cases} \dfrac{dx_1}{dt} = 5x_1 - x_2 \\[2ex] \dfrac{dx_2}{dt} = 3x_1 + x_2 \end{cases}$$

want!

$$\begin{cases} r a_1 e^{rt} = 5a_1 e^{rt} - a_2 e^{rt} \\ r a_2 e^{rt} = 3a_1 e^{rt} + a_2 e^{rt} \end{cases}$$

$$\begin{cases} r a_1 = 5a_1 - a_2 \\ r a_2 = 3a_1 + a_2 \end{cases}$$

$$\begin{cases} 0 = (5-r)a_1 - a_2 \\ 0 = 3a_1 + (1-r)a_2 \end{cases}$$

Hmmm. Don't want $\vec{a} = \vec{0}$ sol$^n$.
If Cramer's rule works, get only the zero sol$^n$:

$$a_1 = \frac{\det\begin{bmatrix} 0 & \ast \\ 0 & \ast \end{bmatrix}}{\det\begin{bmatrix} \ast & \ast \\ \ast & \ast \end{bmatrix}} = 0$$

Must have

$$\det\begin{bmatrix} 5-r & -1 \\ 3 & 1-r \end{bmatrix} = 0$$

$$(5-r)(1-r) - (-1)3 = 0$$

$$r^2 - 6r + 8 = 0$$

$$(r-4)(r-2) = 0$$

$$\vec{x}\,' = A\vec{x}$$

$$r\vec{a}e^{rt} = (A\vec{a})e^{rt}$$

$$\boxed{r\vec{a} = A\vec{a}}$$

$r$ eigenvalue
$\vec{a}$ eigenvector

$$A\vec{a} - r\vec{a} = \vec{0}$$

$$(A - rI)\vec{a} = 0 \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\det(A - rI) = 0}$$

solutions $r$ are e.vals

$r = 2, 4$

Next, find $\vec{a}$ that goes with each r.

$$\begin{cases} 0 = (5-r)a_1 - a_2 \\ 0 = 3a_1 + (1-r)a_2 \end{cases} \qquad \Big| \qquad (A - r\mathbb{I})\vec{a} = 0$$

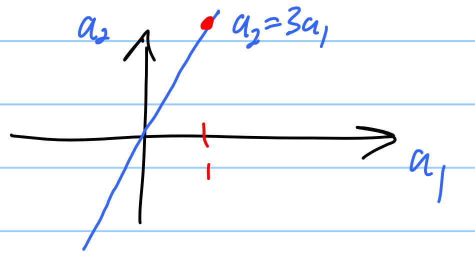
Case $r=2$:

$$\begin{cases} (5-2)a_1 - a_2 = 0 \\ 3a_1 + (1-2)a_2 = 0 \end{cases}$$

$$\begin{cases} 3a_1 - a_2 = 0 \\ 3a_1 - a_2 = 0 \end{cases} \leftarrow \text{sometimes eqn } 2 = \text{const}\cdot(\text{eqn } 1)$$

One eqn: $3a_1 - a_2 = 0 \leftarrow \infty$ many sol$^n$s $\qquad \underline{\underline{a_2 = 3a_1}}$

a line of sol$^n$s.




Take $a_1 = 1$

$a_2 = 3\cdot 1 = 3$

Get one non-zero $\vec{a}$ that works: $\vec{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Found sol$^n$ $\vec{x} = \vec{a}e^{rt} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}$

Case $r=4$: Get $\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$

Superposition principle: If $\vec{x}_1$ and $\vec{x}_2$ solve homog linear system $\vec{x}' = A\vec{x}$, so does $c_1\vec{x}_1 + c_2\vec{x}_2$.

Why: $\vec{x}_1' = A\vec{x}_1$

$\vec{x}_2' = A\vec{x}_2$

Let $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$.

Then $\vec{x}' = c_1\vec{x}_1' + c_2\vec{x}_2'$

$$= c_1 A\vec{x}_1 + c_2 A\vec{x}_2$$

$$\vec{x}\,' = A[\underbrace{c_1\vec{x}_1 + c_2\vec{x}_2}_{\vec{x}}] \quad \checkmark$$

Big question: Is $c_1\begin{pmatrix}1\\3\end{pmatrix}e^{2t} + c_2\begin{pmatrix}1\\1\end{pmatrix}e^{4t}$ the gen$^l$ sol$^n$ to my system? E & U Thm plus superposition princ tells me yes, if I can solve any IVP $\vec{x}(t_0) = \begin{pmatrix}A\\B\end{pmatrix}$.

Check: $$\vec{x}(t_0) = c_1\begin{pmatrix}1\\3\end{pmatrix}e^{2t_0} + c_2\begin{pmatrix}1\\1\end{pmatrix}e^{4t_0} \overset{\text{want}}{=} \begin{pmatrix}A\\B\end{pmatrix}$$

$$\begin{cases} c_1 e^{2t_0} + c_2 e^{4t_0} = A \\ c_1 3e^{2t_0} + c_2 e^{4t_0} = B \end{cases}$$

If Cramer's Rule works, get $c_1, c_2$ no matter what $A, B$ are! $\longrightarrow$ $$\underbrace{\begin{bmatrix} e^{2t_0} & e^{4t_0} \\ 3e^{2t_0} & e^{4t_0} \end{bmatrix}}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}A\\B\end{pmatrix}$$

$$\det = e^{6t_0} - 3e^{6t_0} = \underbrace{-2e^{6t_0}}_{\text{never zero!}}$$

Need $\det[\quad] \neq 0$

Yes, we have gen$^l$ sol$^n$!

Bad news. Get double root $r$. Or complex roots.

**Def$^n$:** For $2\times2$ system $\vec{x}\,' = A$. $\vec{x}_1, \vec{x}_2$ sol$^n$s.

$X = [\vec{x}_1, \vec{x}_2]$ with $\det X \neq 0$, then

$X$ is a Fundamenta Matrix.

$W = W[\vec{x}_1, \vec{x}_2] = \det X$ is the Wronskian.

$$\mathbb{X} = \left[ \begin{pmatrix}1\\3\end{pmatrix} e^{2t}, \begin{pmatrix}1\\1\end{pmatrix} e^{4t} \right] = \begin{bmatrix} e^{2t} & e^{4t} \\ 3e^{2t} & e^{4t} \end{bmatrix}$$

Gen$^l$ Sol$^n$ to $\vec{x}\,' = \mathbb{A}$ is $\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$

$$\vec{x} = \mathbb{X} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbb{X}\, \vec{c}$$

Next time: Complex e.vals $r = a \pm bi$

Get complex e.vect $\vec{a}$ for $r = a + bi$.

Then $\vec{x} = \vec{a}\, e^{(a+bi)t}$ is a complex sol$^n$.

Aha! Real and imag parts are vector sol$^n$s too!

Wronskian is $\neq 0$. Get real gen$^l$ sol$^n$!