Lesson 31 Systems (cont'd)

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 - 3x_2 \\ \dfrac{dx_2}{dt} = -2x_1 + 2x_2 \end{cases}$$

$$\vec{x}' = \mathbb{A}\vec{x}$$

$$\mathbb{A} = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix}$$

Try $\vec{x} = \vec{a}e^{rt}$. Need r = eigenvalue of $\mathbb{A}$

$\vec{a}$ = corresponding eigenvector

How to find r: $\boxed{\det(\mathbb{A} - r\mathbb{I}) = 0}$

$\mathbb{A}$ 2x2 quadratic in r.

($n\times n$, deg n in r)

$$\det \begin{bmatrix} 1-r & -3 \\ -2 & 2-r \end{bmatrix} = 0$$

$$(1-r)(2-r) - (-2)(-3) = 0$$

$$\boxed{r^2 - 3r - 4 = 0} \leftarrow \text{characteristic eqn}$$

$$(r-4)(r+1) = 0 \qquad \text{e.vals: } r = -1, 4$$

How to find $\vec{a}$ for e.val r: $\boxed{(\mathbb{A} - r\mathbb{I})\vec{a} = \vec{0}}$

For $r=4$: $(\mathbb{A} - 4\mathbb{I})\vec{a} = \vec{0}$

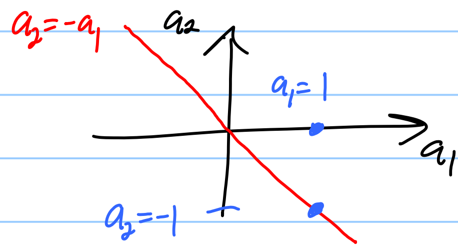
$$\begin{bmatrix} 1-4 & -3 \\ -2 & 2-4 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} -3 & -3 \\ -2 & -2 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{matrix} \leftarrow -\frac{1}{3}(\text{eqn } 1) \\ \leftarrow (\text{Row } 2) - \frac{2}{3}(\text{Row } 1) \end{matrix}$$

$$\begin{cases} -3a_1 - 3a_2 = 0 \\ -2a_1 - 2a_2 = 0 \end{cases} \rightarrow a_1 + a_2 = 0 \qquad \boxed{a_2 = -a_1}$$

∞ many sol^{n}s!



Just need one non-zero $\vec{a}$,

Take $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Get a sol^n to $\vec{x}' = \mathbb{A}\vec{x}$: $\quad \vec{x} = \vec{a}e^{rt} = \underline{\underline{\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}}}$

For $r=-1$: $\quad (\mathbb{A} - (-1)\mathbb{I})\vec{a} = \vec{0}$

$$\left[\begin{array}{cc|c} 1-(-1) & -3 & 0 \\ -2 & 2-(-1) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & -3 & 0 \\ -2 & 3 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] \leftarrow (\text{Row } 2) + 1 \cdot (\text{Row } 1)$$

Quick trick: $\quad \left[\begin{array}{cc|c} A & B & 0 \\ 0 & 0 & 0 \end{array}\right] \leftarrow$ eyeball a non-zero sol^n

$$\underline{\underline{\begin{pmatrix} B \\ -A \end{pmatrix} \text{ or } \begin{pmatrix} -B \\ A \end{pmatrix}}}$$

$$Aa_1 + Ba_2 = 0$$

$$a_2 = -\frac{A}{B}a_1$$

$a_1 = 1$: $\quad \begin{pmatrix} 1 \\ -A/B \end{pmatrix} \qquad a_1 = B$: $\quad \begin{pmatrix} B \\ -A \end{pmatrix}$ ✓

$a_1 = -B$: $\begin{pmatrix} -B \\ A \end{pmatrix}$ ✓

$$\left[\begin{array}{cc|c} 2 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Get $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \vec{a}$.

(All possible $\vec{a}$'s : $c\begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $c \neq 0$)

Get sol^n: $\vec{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t}$.

Gen^l Sol^n: $\vec{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t}$

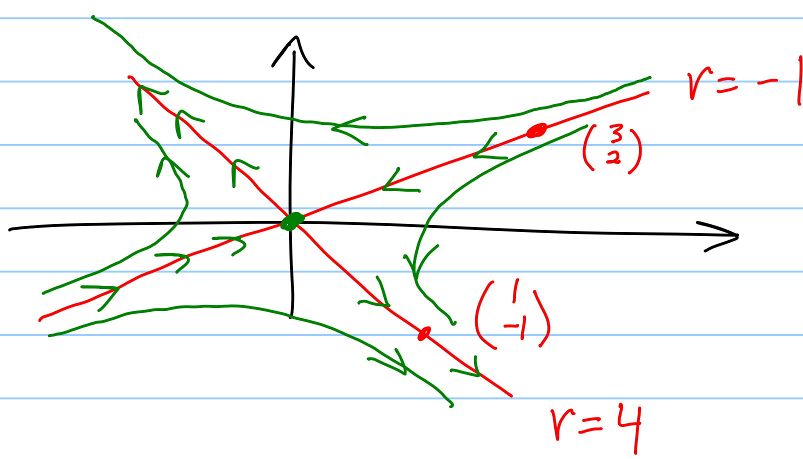
Because $\mathbb{X} = \begin{bmatrix} e^{4t} & 3e^{-t} \\ -e^{4t} & 2e^{-t} \end{bmatrix}$ ← Fundamental matrix

$\uparrow \vec{x}_1$ $\quad$ $\uparrow \vec{x}_2$

Wronskian: $W = \det[\vec{x}_1, \vec{x}_2] = 2e^{3t} - (-3)e^{3t} = 5e^{3t}$

↑ never zero!

$W \neq 0$ means that we can satisfy any and all IVPs.



$\begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{-t}$

$\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{4t}$

(0,0) is a critical point called a <u>saddle point</u>. e-vals real, opposite sign.

Complex e-vals:

$$\begin{cases} \dfrac{dx_1}{dt} = -2x_1 + 2x_2 \\[2ex] \dfrac{dx_2}{dt} = -2x_1 - 2x_2 \end{cases} \qquad A = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix}$$

$$\det\begin{pmatrix} -2-r & 2 \\ -2 & -2-r \end{pmatrix} = (-2-r)^2 + 4 = 0$$

$$r^2 + 4r + 8 = 0$$

$$r = \frac{-4 \pm \sqrt{16-32}}{2} = \underline{\underline{-2 \pm 2i}} \quad \leftarrow \text{pretend not to notice!}$$

For $r = -2+2i$:

$$\left[\begin{array}{cc|c} -2-(-2+2i) & 2 & 0 \\ -2 & -2-(-2+2i) & 0 \end{array}\right]$$

$$\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \left[\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right] \end{array} \leftarrow \text{Row 2} = -i\,(\text{Row 1})$$

$$\sim \left[\begin{array}{cc|c} A & B & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \vec{a} = \begin{pmatrix} B \\ -A \end{pmatrix} = \begin{pmatrix} 2 \\ 2i \end{pmatrix}$$

Even beter $\vec{a} = \frac{1}{2}\begin{pmatrix} 2 \\ 2i \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

Get complex sol^n: $\vec{x} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-2+2i)t}$

$$= \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]\left(e^{-2t}\cos 2t + i\, e^{-2t}\sin 2t\right)$$

$$= \Bigg[\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t}\cos 2t - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}\sin 2t}_{\vec{x}_1}\Bigg] + i\Bigg[\underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t}\cos 2t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t}\sin 2t}_{\vec{x}_2}\Bigg]$$

Real and imaginary parts are <u>real</u> sol^{n}s.

Gen^l Sol^n $\vec{x} = c_1 \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^{-2t}$

$W \neq 0$. Gen^l Sol^n