Lesson 33 2x2 linear homogeneous systems with constant coefficients

Vectors in WebAssign: $\vec{a}e^{rt} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{rt}$ = <a_1, a_2>e^(rt) in WebAssign

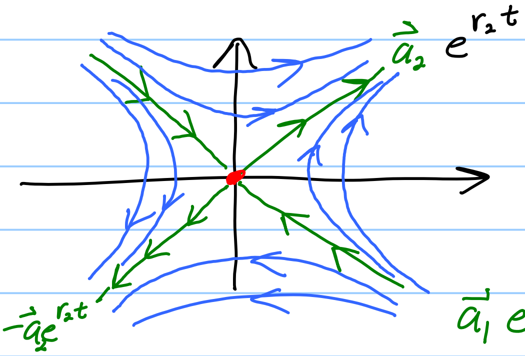
$$\begin{cases} \dfrac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 \\ \dfrac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 \end{cases} \qquad \vec{x}' = \mathbb{A}\vec{x} \qquad \text{where } \mathbb{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \ \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Try $\vec{x} = \vec{a}e^{rt}$. Need $\underbrace{(\mathbb{A} - r\mathbb{I})}_{\text{Need det}=0}\vec{a} = \vec{0}$

Characteristic Eqn: $\det(\mathbb{A} - r\mathbb{I}) = 0$

Cases 1) r_1, r_2 real and opposite sign. $r_1 < 0 < r_2$

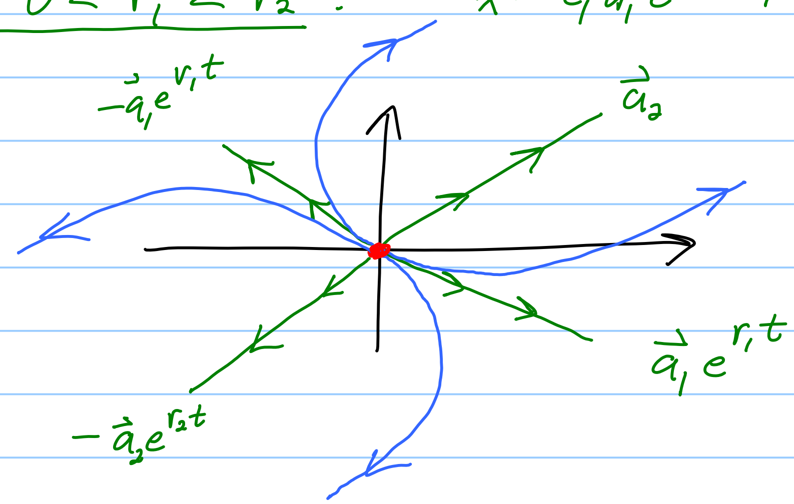
Gen^l Sol^n: $\vec{x} = c_1\vec{a}_1 e^{r_1 t} + c_2\vec{a}_2 e^{r_2 t}$



Type: Saddle Point
Stability: Unstable

Case 2: r_1, r_2 real and same sign and $\neq$.

Subcase $0 < r_1 < r_2$ (closest to zero): $\vec{x} = c_1\vec{a}_1 e^{r_1 t} + c_2\vec{a}_2 e^{r_2 t}$



Node rule: Trajectories hug the ± e.vector near the origin corresponding to the e-val closest to zero

Node rule part 2: Far from the origin, trajectories approach looking ∥ to other ± e.vect.

Subcase $r_2 < r_1 < 0$: closest to zero — Same picture! But arrows go in.

Type: Improper node.

Stability: Positive r's: Unstable

Negative r's: Asymptotically Stable

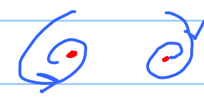
Case $r = a \pm bi$ complex : Get complex e. vect for $r = a + bi$.

$$(\vec{A} + \vec{B}i)\, e^{(a+bi)t} = \underbrace{\left[\vec{A}\cos bt - \vec{B}\sin bt\right] e^{at}}_{\vec{x}_1} + i \underbrace{\left[\vec{A}\sin bt + \vec{B}\cos bt\right] e^{at}}_{\vec{x}_2}$$

Critical thing: Sign of a.

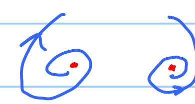
Subcase $a < 0$: Type: Spiral (in)

Stability: Asympt. Stable



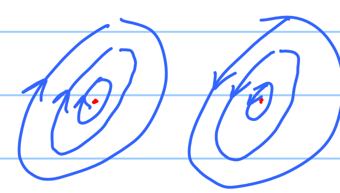
Subcase $a > 0$: Type: Spiral (out)

Stability: Unstable



Subcase $a = 0$ Type: Center

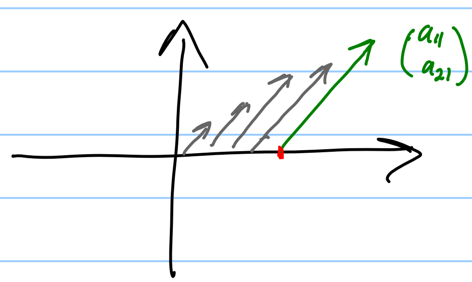
Stability: Stable



CW or CCW? Test field at $(1, 0)$

$$\vec{x}' = A\vec{x}. \quad \text{So} \quad \vec{x}'\Big|_{\text{at } (1,0)} = A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a_{11}\\ a_{21}\end{pmatrix}$$

first column of A



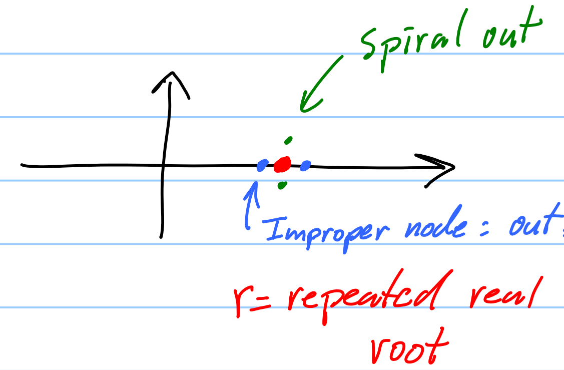
Spiral rule: $a_{21} > 0$: CCW

$a_{21} < 0$: CW

What about $r_1 = r_2$ cases or r's $=0$ cases?

Very delicate and engineers avoid them.

Quadratic eqn: $r = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$



3X3 case:

$$\begin{cases} \dfrac{dx_1}{dt} = 4x_2 \\[2mm] \dfrac{dx_2}{dt} = -x_1 \\[2mm] \dfrac{dx_3}{dt} = x_1 + 4x_2 - x_3 \end{cases} \qquad \vec{x}\,' = \begin{bmatrix} 0 & 4 & 0 \\ -1 & 0 & 0 \\ 1 & 4 & -1 \end{bmatrix} \vec{x}$$

$$\det(A - r\mathbb{I}) = \det \begin{bmatrix} 0-r & 4 & 0 \\ -1 & 0-r & 0 \\ 1 & 4 & -1-r \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= +(-1-r)\det\begin{bmatrix} -r & 4 \\ -1 & -r \end{bmatrix} - 0\det + 0\det$$

$$= -(r+1)(r^2+4)$$

Roots: $-1, \pm 2i$

For $r=1$: $(A-(-1)\mathbb{I})\vec{a} = \vec{0}$

$$\left[\begin{array}{ccc|c} 0-(-1) & 4 & 0 & 0 \\ -1 & 0-(-1) & 0 & 0 \\ 1 & 4 & -1-(-1) & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \leftarrow a_1 + 4a_2 = 0 \\ \leftarrow -5a_2 = 0 \quad \boxed{a_2 = 0} \end{array}$$

So $a_2 = 0$.

$a_1 = -4a_2 = -4 \cdot 0 = 0$

$\boxed{a_1 = 0}$

$a_3 =$ anything! $\quad \vec{a} = \begin{pmatrix} 0 \\ 0 \\ a_3 \end{pmatrix} \quad$ Take $a_3 = 1$.

Get $\vec{a} e^{-t} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-t}$ as one sol^n.

<u>For $r = 2i$</u>: Get complex e-vect. Split into <u>two</u> real sol^{n}s.

Gen sol^n: $\quad \vec{x} = c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -2\cos 2t \\ -\sin 2t \\ 2\cos 2t \end{pmatrix} + c_3 \begin{pmatrix} -2\sin 2t \\ \cos 2t \\ 2\sin 2t \end{pmatrix}$