MA 266 Exam 2 solutions – Bell

(1.) $r^2 + 4r + 5 = 0$

$r = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$

Ans: $y = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t$

(2.) $r^2 + 5r + 6 = 0$

$(r+2)(r+3) = 0$

$r = -2, -3$

Homog Sol^n: $y = c_1 e^{-2t} + c_2 e^{-3t}$

Part. Sol^n: $y_p = Ae^{-t}$

$y_p' = -Ae^{-t}$

$y_p'' = Ae^{-t}$

Need $\underbrace{[Ae^{-t}]}_{y_p''} + 5\underbrace{[-Ae^{-t}]}_{y_p'} + 6\underbrace{[Ae^{-t}]}_{y_p} = \overset{\text{want}}{e^{-t}}$

$\underbrace{(A - 5A + 6A)}_{=1} e^{-t} = e^{-t}$ $\quad 2A = 1$

$\boxed{A = 1/2}$

Gen^l Sol^n: $y = c_1 e^{-2t} + c_2 e^{-3t} + \frac{1}{2} e^{-t}$

$y' = -2c_1 e^{-2t} - 3c_2 e^{-3t} - \frac{1}{2} e^{-t}$

$$\begin{cases} y(0) = c_1 + c_2 + \frac{1}{2} = 0 \\ y'(0) = -2c_1 - 3c_2 - \frac{1}{2} = 0 \end{cases}$$

want

$$\begin{cases} c_1 + c_2 = -1/2 \\ -2c_1 - 3c_2 = 1/2 \end{cases}$$

$\leftarrow 2c_1 + 2c_2 = -1$

$+$

$-c_2 = -1/2$

$c_1 = -\frac{1}{2} - c_2 = -1$

$\boxed{c_1 = -1 \quad c_2 = \frac{1}{2}}$

Ans: $y = -e^{-2t} + \frac{1}{2} e^{-3t} + \frac{1}{2} e^{-t}$

(3.) $r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r = -1, -1$

Homog sol^n: $y = c_1 e^{-t} + c_2 t e^{-t}$

Part sol^n: $y_p = A \cos 2t + B \sin 2t$

$y_p' = -2A \sin 2t + 2B \cos 2t$

$y_p'' = -4A \cos 2t - 4B \sin 2t$

Need $[-4A\cos 2t - 4B\sin 2t] + 2[-2A \sin 2t + 2B\cos 2t] + [A\cos 2t + B \sin 2t] = \overset{\text{want}}{\sin 2t}$

$\underbrace{(-3A + 4B)}_{0} \cos 2t + \underbrace{(-4A - 3B)}_{1} \sin 2t = \sin 2t$

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$$\begin{bmatrix} -4 & -3 \\ -3 & 4 \end{bmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A = \frac{\det \begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix}}{\det \begin{bmatrix} -4 & -3 \\ -3 & 4 \end{bmatrix}} = \frac{4}{(-25)} = -\frac{4}{25}$$

$$B = \frac{\det \begin{bmatrix} -4 & 1 \\ -3 & 0 \end{bmatrix}}{\det \begin{bmatrix} -4 & -3 \\ -3 & 4 \end{bmatrix}} = \frac{3}{(-25)} = -\frac{3}{25}$$

Ans: $y = c_1 e^{-t} + c_2 t e^{-t} - \frac{4}{25}\cos 2t - \frac{3}{25}\sin 2t$

4. Standard Form: $y'' + \frac{3}{t} y' - \frac{3}{t^2} y = \underbrace{\frac{16}{t}}_{F(t)}$

$$W[y_1, y_2] = \det \begin{bmatrix} t & t^{-3} \\ 1 & -3t^{-4} \end{bmatrix} = -3t^{-3} - t^{-3} = -4t^{-3}$$

$y_1 = t$ $\quad y_2 = t^{-3}$

$y_p = u_1 y_1 + u_2 y_2$ where

$$u_1' = \frac{-y_2 F}{W} = \frac{-t^{-3}\left(\frac{16}{t}\right)}{(-4t^{-3})} = \frac{4}{t}$$

So $\boxed{u_1 = 4\ln t}$

$$u_2' = \frac{y_1 F}{W} = \frac{t\left(\frac{16}{t}\right)}{(-4t^{-3})} = -4t^3$$

So $\boxed{u_2 = -t^4}$

Ans: $y = (4\ln t)\,t + (-t^4)\,t^{-3} = 4t\ln t - t$

solves homog

So $y_p = 4t\ln t$ works too!

5. $y = x^r$

$y' = r x^{r-1}$

$y'' = r(r-1)x^{r-2}$

$y''' = r(r-1)(r-2)x^{r-3}$

plug in ode: $x^3\left[r(r-1)(r-2)x^{r-3}\right] - 12x\left[r x^{r-1}\right] \overset{\text{want}}{=} 0$

$$\underbrace{\left[r(r-1)(r-2) - 12r\right]}_{\text{need } = 0} x^r = 0$$

$r\left[(r-1)(r-2) - 12\right] = 0$

$r\left(r^2 - 3r - 10\right) = 0$

$r(r-5)(r+2) = 0$ $\quad r = 0, 5, -2$

Get $\begin{cases} y_1 = x^0 = 1 \\ y_2 = x^5 \\ y_3 = \frac{1}{x^2} \end{cases}$

Expect Gen^{l} Sol^{n} for this linear homog eqn to be

$$\underline{y = c_1 + c_2 x^5 + c_3 \cdot \frac{1}{x^2}}$$

Check Wronskian:

$$W = \det \begin{bmatrix} 1 & x^5 & x^{-2} \\ 0 & 5x^4 & -2x^{-3} \\ 0 & 20x^3 & 6x^{-4} \end{bmatrix} = 30 - (-40) = 70$$

not zero!

So yes, this is the gen^{l} sol^{n}.

6. $r = 0, \underbrace{1, 1}_{e^t,\, te^t}, -1$

$e^{0 \cdot t} = 1$ e^{-t}

Homog sol^{n}: $y = c_1 + c_2 e^t + c_3 te^t + c_4 e^{-t}$

FORM of $y_p = t(A_2 t^2 + A_1 t + A_0) + t^2(B_1 t + B_0)e^t$

$y = A$ solves homog

Be^t, Bte^t solve homog

7. $r^4 - 16 = 0$

$(r^2 - 4)(r^2 + 4) = 0$

$(r-2)(r+2)(r^2+4) = 0$

$r = \pm 2, \pm 2i$

Gen^{l} Sol^{n} $= \underline{c_1 e^{-2t} + c_2 e^{2t} + c_3 \cos 2t + c_4 \sin 2t}$