MA 266 Practice problems for Exam 2 sol^{n}s

On which interval is the initial value problem

$$\begin{cases} (5-t)y'' + (t-4)y' + 2y = \ln t \\ y(2) = 8 \end{cases}$$

guaranteed to have a unique solution?

Standard Form: $y'' + \underbrace{\frac{t-4}{t-5}}_{P(t)} y' + \underbrace{\frac{1}{t-5}}_{Q(t)} y = \underbrace{\frac{\ln t}{t-5}}_{R(t)}$

The largest interval containing $x_0 = 2$ on which $P(t)$, $Q(t)$, and $R(t)$ are all continuous is $(0,5)$.

Find the general solution of the differential equation

$$y'' - 10y' + 27y = 0$$

$r^2 - 10r + 27 = 0$ Roots: $r = \frac{10 \pm \sqrt{100-108}}{2}$

$= 5 \pm i\sqrt{2}$

$y = c_1 e^{5t} \cos\sqrt{2}\,t + c_2 e^{5t} \sin\sqrt{2}\,t$

The function $y_1 = e^x$ is a solution of

$$(x-1)y'' - 2xy' + (x+1)y = 0.$$

If we seek a second solution $y_2 = e^x v(x)$ by reduction of order, then $v(x) =$

Standard Form: $y'' + \underbrace{\left[\frac{-2x}{x-1}\right]}_{P(x)} y' + \left[\frac{x+1}{x-1}\right] y = 0$

$y_1 = e^x$ is one sol^n.

$y_2 = v\,y_1$ where $v' = \frac{1}{y_1^2} e^{-\int P(x)\,dx}$

$$v' = \frac{1}{(e^x)^2}\, e^{-\int -\frac{2x}{x-1}\,dx}$$

$$= e^{-2x}\, e^{2\int \frac{x-1+1}{x-1}\,dx}$$ ← or do long division $x-1\overline{)\,x}$

$$= e^{-2x}\, e^{2\int 1 + \frac{1}{x-1}\,dx}$$

$$v' = e^{-2x}\, e^{2(x + \ln|x-1|)} = e^{\ln|x-1|^2} = (x-1)^2$$

So $v = \int (x-1)^2\,dx = \underline{\frac{1}{3}(x-1)^3}$ and

$$y_2 = v\, y_1 = \underline{\frac{1}{3}(x-1)^3 e^x}$$

By the Method of Undetermined Coefficients, which of the following Y is the correct form of a particular solution to the equation

$$L[y] = y^{(4)} - 3y^{(3)} + 2y^{(2)} = 4t - e^t + 3e^{3t}?$$

$F_1(t) = 4t$
$F_2(t) = -e^t$
$F_3(t) = 3e^{3t}$

Homog sol^n: $r^4 - 3r^3 + 2r^2 = 0$

$r^2(r^2 - 3r + 2) = 0$

$r^2(r-2)(r-1) = 0$

$$\overbrace{e^{0t}}^{1} \quad \overbrace{te^{0t}}^{t} \quad e^t \quad e^{2t}$$

$r = 0, 0, 1, 2$

Gen^l sol^n homog: $y = c_1 + c_2 t + c_3 e^t + c_4 e^{2t}$

Particular sol^n $y_p = y_{p_1} + y_{p_2} + y_{p_3}$ where $L[y_{p_j}] = F_j$

$$y_p = \underbrace{t^2[A_1 t + A_0]}_{\text{because } 1,t \text{ solve homog}} + \underbrace{t[Be^t]}_{\text{because } e^t \text{ solves homog}} + [Ce^{3t}]$$

Which of the following is the correct form of a particular solution to the equation

$$L[y] = y^{(4)} - y = \underline{2 + 3te^{-t} + 2\sin t} \quad ?$$

$$\begin{cases} F_1(t) = 2 \\ F_2(t) = 3te^{-t} \\ F_3(t) = 2\sin t \end{cases}$$

Homog sol^n: $r^4 - 1 = 0$

$$(r^2-1)(r^2+1) = 0$$

$$(r-1)(r+1)(r^2+1) = 0 \qquad r = 1, -1, \underbrace{\pm i}_{\cos t,\ \sin t}$$

$e^t \quad e^{-t}$

$$y = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$$

$$y_p = y_{p_1} + y_{p_2} + y_{p_3} \quad \text{where} \quad L[y_{p_j}] = F_j(t)$$

$$y_p = [A] + \underbrace{t[(B_1 t + B_0)e^{-t}]}_{\text{because } Be^{-t} \text{ solves homog}} + \underbrace{t[C\cos t + D\sin t]}_{\text{because } \cos t \text{ and } \sin t \text{ solve homog}}$$

Find the solution of the initial value problem

$$y'' + y' - 6y = 0, \quad y(0) = 0, \quad y'(0) = 5.$$

$$r^2 + r - 6 = 0$$

$$(r+3)(r-2) = 0 \qquad r = 2, -3$$

$$y = c_1 e^{2x} + c_2 e^{-3x}$$

$$y' = 2c_1 e^{2x} - 3c_2 e^{-3x}$$

want

$$\begin{cases} y(0) = c_1 + c_2 = 0 \leftarrow c_2 = -c_1 \\ y'(0) = 2c_1 - 3c_2 = 5 \end{cases}$$

$$2c_1 - 3(-c_1) = 5 \qquad \boxed{\begin{matrix} c_1 = 1 \\ c_2 = -1 \end{matrix}}$$

$$\underline{y = e^{2x} - e^{-3x}}$$

A spring-mass system set in motion was determined by the initial value problem

$$u'' + 100u = 0, \quad u(0) = 1, \quad u'(0) = -10.$$

What is the amplitude of the motion?

$r^2 + 100 = 0$, $r = \pm 10i$

$u = c_1 \cos 10t + c_2 \sin 10t$

$u' = -10c_1 \sin 10t + 10c_2 \cos 10t$

$u(0) = c_1 = 1$ want

$u'(0) = 10c_2 = -10$

$$\boxed{\begin{array}{l} c_1 = 1 \\ c_2 = -1 \end{array}}$$

$$u = c_1 \cos 10t + c_2 \sin 10t$$

$$= A \cos(10t - \varphi)$$

where $A = \text{Amplitude} = \sqrt{c_1^2 + c_2^2} = \sqrt{(1)^2 + (-1)^2} = \underline{\sqrt{2}}$

An undamped, free vibration $u'' + 9u = 0$ has initial conditions $u(0) = 4$, $u'(0) = 9$. The solution of this initial value problem can be written as $u = R\cos(\omega t - \delta)$. What are R and δ?

$u = c_1 \cos 3t + c_2 \sin 3t$

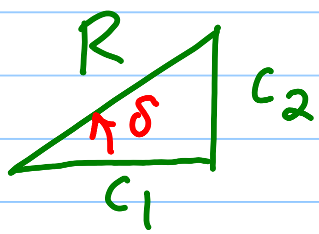
$u' = -3c_1 \sin 3t + 3c_2 \cos 3t$

want

$u(0) = c_1 = 4$

$u'(0) = 3c_2 = 9$

$$\boxed{\begin{array}{l} c_1 = 4 \\ c_2 = 3 \end{array}}$$



$$u = c_1 \cos 3t + c_2 \sin 3t$$

$$= \underbrace{\sqrt{c_1^2 + c_2^2}}_{R} \left(\underbrace{\frac{c_1}{\sqrt{c_1^2 + c_2^2}}}_{\cos\delta} \cos 3t + \underbrace{\frac{c_2}{\sqrt{c_1^2 + c_2^2}}}_{\sin\delta} \sin 3t \right)$$

$= R \cos(3t - \delta)$ where $R = \sqrt{4^2 + 3^2} = \underline{\underline{5}}$

and $\delta = \underline{\tan^{-1} \frac{3}{4}}$

A spring system with external forcing term is represented by the equation

$$y'' + 2y' + 2y = 4\cos(t) + 2\sin(t).$$

Then the steady state solution of the system is given by

Homog sol^n is $c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$ is transient.

Steady state sol^n = $y_p = A\cos t + B\sin t$

$$y_p' = -A\sin t + B\cos t$$

$$y_p'' = -A\cos t - B\sin t$$

Plug into ODE: $[-A\cos t - B\sin t] + 2[-A\sin t + B\cos t] + 2[A\cos t + B\sin t]$

$$= \underbrace{[-A + 2B + 2A]}_{\text{want } = 4}\cos t + \underbrace{[-B - 2A + 2B]}_{=2}\sin t$$

$$\begin{cases} A + 2B = 4 \\ -2A + B = 2 \end{cases}$$

$$\begin{aligned} 2A + 4B &= 8 \\ -2A + B &= 2 \\ \hline 5B &= 10 \\ B &= 2 \end{aligned}$$

$A = 4 - 2B$
$= 4 - 2\cdot 2 = 0$
$A = 0$

Steady state sol^n = $0\cdot\cos t + 2\cdot\sin t = 2\sin t$

The homogeneous differential equation $t^2y'' - 4ty' + 6y = 0$ $(t > 0)$ has two solutions given by $y_1(t) = t^2$ and $y_2(t) = t^3$. Using the method of Variation of Parameters, find the general solution of the nonhomogeneous equation $t^2y'' - 4ty' + 6y = t^3$.

Standard Form! $y'' + \left[\frac{-4}{t}\right]y' + \left[\frac{6}{t^2}\right]y = \underbrace{t}_{F(t)}$

$$W = W[y_1, y_2] = \det \begin{bmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{bmatrix} = 3t^4 - 2t^4 = t^4$$

$y_p = u_1 y_1 + u_2 y_2$ where

$u_1' = \dfrac{-y_2 F}{W}$	$u_2' = \dfrac{y_1 F}{W}$
$u_1' = \dfrac{-t^3 \cdot t}{t^4} = -1$	$u_2' = \dfrac{t^2 \cdot t}{t^4} = \dfrac{1}{t}$
So $\underline{u_1 = -t}$	$\underline{u_2 = \ln\lvert t\rvert}$

$$y_p = u_1 y_1 + u_2 y_2 = (-t)\cdot t^2 + (\ln|t|)\cdot t^3$$

Gen^l sol^n to $(+)$: $y = \underline{c_1 t^2 + c_2 t^3 + (-1 + \ln|t|)\,t^3}$

Find the general solution of

$$y^{(4)} - 10y'' + 9y = 0.$$

$$r^4 - 10r^2 + 9 = 0$$

$$(r^2 - 9)(r^2 - 1) = 0$$

$$(r-3)(r+3)(r-1)(r+1) = 0 \qquad r = \pm 1, \pm 3$$

$$y = \underline{c_1 e^{x} + c_2 e^{-x} + c_3 e^{3x} + c_4 e^{-3x}}$$

$\underline{\text{or}}$ $\quad C_1 \cosh x + C_2 \sinh x + C_3 \cosh 3x + C_4 \sinh x$

Given that the function $y_1 = t$ is a solution to the differential equation

$$t^2 y'' - ty' + y = 0, \quad t > 0,$$

choose a function y_2 from the list below so that the pair $\{y_1, y_2\}$ is a fundamental set of the solutions to the differential equation above.

Standard Form! $y'' + \underbrace{\left[-\frac{1}{t}\right]}_{P(t) = -\frac{1}{t}} y' + \left[\frac{1}{t^2}\right] y = 0$

$y_2 = v\, y_1$ where

$$v' = \frac{1}{y_1^2} e^{-\int P(t)\,dt}$$

$$= \frac{1}{t^2} e^{-\int -\frac{1}{t}\,dt}$$

$$= \frac{1}{t^2} e^{\ln|t|} = \frac{|t|}{t^2} = \pm \frac{1}{t}$$

take $v' = +\frac{1}{t}$

$$v = \int \frac{1}{t}\,dt = \ln|t|$$

$$y_2 = v\, y_1 = (\ln|t|) \cdot t$$

$$\{y_1, y_2\} = \underline{\{t, t \ln|t|\}}.$$