MATH 341, Exam 1

Each problem is 20 points

(20) **1.** Find

$$\lim_{n\to\infty}\left(\sqrt{n^2+n} - \sqrt{n^2+1}\right).$$

(20) **2. a)** What are $\lim_{n\to\infty} 2^{1/n}$ and $\lim_{n\to\infty} 3^{1/n}$?

b) Given that $\left(1+\frac{1}{n}\right)^n$ is an increasing sequence of real numbers between 2 and 3 that converges to the famous number e as $n \to \infty$ where $2 < e < 3$, explain how to find

$$\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^n.$$

(20) **3.** Compute

$$\left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^7 + \left(\frac{1}{2}\right)^8 + \cdots$$

b) Let $s_n = 1 + r + r^2 + \cdots + r^n$ denote the partial sums of a geometric series with $0 < r < 1$. Show that (s_n) is a Cauchy sequence. (Start with "*Let* $\epsilon > 0$...")

(20) **4.** Suppose A is a subset of the real numbers that is bounded from above. Define the *supremum* Sup A and state why it exists.

(20) **5.** Prove that $[0,1]$ is *uncountable* via Cantor's argument involving the Nested closed interval theorem.

Solutions to Exam 1

1. $\left(\sqrt{n^2+n}-\sqrt{n^2+1}\right)\cdot\dfrac{\sqrt{n^2+n}+\sqrt{n^2+1}}{\sqrt{n^2+n}+\sqrt{n^2+1}} = \dfrac{n-1}{\sqrt{n^2+n}+\sqrt{n^2+1}}\cdot\dfrac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} =$

$$= \frac{1-\frac{1}{n}}{\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{1}{n^2}}} \longrightarrow \frac{1-0}{\sqrt{1+0}+\sqrt{1+0^2}} = \frac{1}{2}$$

using laws of limits, including $\lim\limits_{n\to\infty}\sqrt{x_n} = \sqrt{\lim\limits_{n\to\infty} x_n}$ when $x_n \geq 0$.

2. a) $\lim\limits_{n\to\infty} 2^{1/n}, 3^{1/n} = 1$

b) $2 < \left(1+\frac{1}{n^2}\right)^{n^2} < 3$

So $2^{1/n} < \underbrace{\left[\left(1+\frac{1}{n^2}\right)^{n^2}\right]^{1/n}}_{\left(1+\frac{1}{n^2}\right)^n} < 3^{1/n}$.

Consequently, $\lim\limits_{n\to\infty}\left(1+\frac{1}{n^2}\right)^n = 1$ by the Squeeze Thm.

3. $\left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6 + \cdots + \left(\frac{1}{2}\right)^N$

$$= \left(\tfrac{1}{2}\right)^5\left[1+\left(\tfrac{1}{2}\right)+\left(\tfrac{1}{2}\right)^2+\cdots+\left(\tfrac{1}{2}\right)^{N-5}\right]$$

$$= \left(\tfrac{1}{2}\right)^5\left[\frac{1-\left(\frac{1}{2}\right)^{N-4}}{1-\frac{1}{2}}\right] \to \left(\tfrac{1}{2}\right)^5\cdot\frac{1}{1-1/2}$$

since $0<\frac{1}{2}<1$. $\underline{\text{Ans}} = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$

b) Let $\varepsilon > 0$. Suppose $n, m \in \mathbb{N}$, $m > n$.

$$s_m - s_n = r^{n+1} + r^{n+2} + \cdots + r^m$$
$$= r^{n+1}\left[1 + r + \cdots + r^{m-(n+1)}\right]$$
$$= r^{n+1} \cdot \frac{1-r^{m-n}}{1-r} < \frac{r^{n+1}}{1-r}$$

Since $0<r<1$, $\lim_{n\to\infty} r^{n+1} = 0$. So there is an $N \in \mathbb{N}$ such that $r^{n+1} < \varepsilon(1-r)$ when $n \geq N$. Hence

$$|s_m - s_n| < \frac{r^{n+1}}{1-r} < \underbrace{\frac{\varepsilon(1-r)}{1-r}}_{=\varepsilon} \quad \text{when} \quad m > n \geq N \text{ and}$$

we have shown that (s_n) is a Cauchy seq.

4. Sup A is the <u>least</u> upper bound of A. It exists because $\mathbb{R}$ is <u>complete</u>.

5. Suppose $[0,1]$ is countable. Then there is a one-to-one mapping $\varphi: \mathbb{N} \to [0,1]$ that is <u>onto</u>. Let $r_n = \varphi(n)$. For r_1, pick a closed interval $[a_1, b_1] \subset [0,1]$ such that $0 \leq a_1 < b_1 \leq 1$ such that $r_1 \notin [a_1, b_1]$. For r_2, pick a closed interval $[a_2, b_2] \subset [a_1, b_1]$ such that $a_1 \leq a_2 < b_2 \leq b_1$ such that $r_2 \notin [a_2, b_2]$. Etc. Get a sequence of closed intervals $[0,1] \supset [a_1, b_1] \supset [a_2, b_2] \supset \cdots$

such that $r_n \notin [a_n, b_n]$ for all n. Now the Nested (closed) interval theorem yields a real number $r \in \bigcap_{n=1}^{\infty} [a_n, b_n]$. But $r \in [a_n, b_n]$ and $r_n \notin [a_n, b_n]$ implies that $r \neq r_n$ for each n. Hence φ is <u>not</u> onto $[0,1]$. This contradiction implies that no such mapping φ exists. So $[0,1]$ is uncountable.