Math 341

Exam 2

Each problem is worth 25 points

1. A set $K \subset \mathbb{R}$ is called *compact* if every sequence (x_n) of elements in K has a subsequence (x_{n_k}) that converges to *a point in* K.

a) Assume $a < b$. Prove that $[a, b]$ is compact

b) Prove that if f is continuous on $\mathbb{R}$ and $K \subset \mathbb{R}$ is compact, then $f(K) = \{f(x) : x \in K\}$ is compact.

2. Find $\lim_{x \to 0+} x^{\sin 3x}$. Explain your work.

3. Suppose that $R > 0$ and $x \in [-R, R]$. Prove that

$$0 < e^x - \sum_{n=0}^{N} \frac{x^n}{n!} < \frac{R^{N+1} e^R}{(N+1)!}.$$

4. Find $F'(x)$ when F is defined on $[0, 1]$ by

$$F(x) = \int_{x^2}^{x} \sqrt{1 + t^7} \; dt.$$

Explain your work.

MA 341 Exam 2 solutions

1. a) We want to show that, given seq (x_n) in $[a,b]$, there is a convergent subseq (x_{n_k}) that converges to $x \in [a,b]$. Since (x_n) is a bounded seq, the Bolzano-Weierstraß theorem yields a subseq (x_{n_k}) that converges, say to x. Since $a \le x_{n_k} \le b$, the Squeeze theorem implies $a \le x \le b$. So $x \in [a,b]$ and $[a,b]$ is compact.

b) Suppose $y_n \in f(K) = \{f(x) : x \in K\}$. Then there exists $x_n \in K$ with $y_n = f(x_n)$. Given a seq (y_n) in $f(K)$, we get a seq (x_n) in K. K compact $\Rightarrow$ there is a convergent subseq (x_{n_k}) with $x_{n_k} \to x \in K$. Since f is continuous on K, the sequential criterion for continuity yields that $f(x_{n_k}) \to f(x)$, which is a point in $f(K)$. Hence $f(K)$ is compact.

2. Assume $x > 0$. Then $x^{\sin 3x} = e^{(\sin 3x)\ln x}$.

$$(\sin 3x)\ln x = \frac{\ln x}{\left(\frac{1}{\sin 3x}\right)} \sim \frac{-\infty}{\infty} \text{ as } x \to 0+$$

Use L'Hôpital's:
$$\lim_{x\to 0+} \frac{\ln x}{\left(\frac{1}{\sin 3x}\right)} \overset{L'H}{=} \lim_{x\to 0+} \frac{\left(\frac{1}{x}\right)}{\left(\frac{3\cos 3x}{\sin^2 3x}\right)} =$$

$$= \lim_{x\to 0+} \frac{3\cdot \sin 3x}{3x} \cdot \frac{1}{3} \cdot \frac{\sin 3x}{\cos 3x} = 3 \cdot \frac{1}{3} \cdot \left(\frac{0}{1}\right) = 0. \quad \left[\frac{\sin\theta}{\theta} \to 1 \text{ as } \theta \to 0\right.$$

Since e^t is continuous at $t=0$, we obtain

$$x^{\sin 3x} = e^{(\sin 3x)\ln x} \to e^0 = 1 \text{ as } x \to 0+.$$

3. Taylor's with remainder yields

$$e^x = \frac{d^{(N+1)}}{dx^{N+1}} e^x \qquad \frac{f^{(N+1)}(c)}{(N+1)!}(x-x_0)^{N+1}$$

$$R = e^x - \sum_{n=0}^{N} \frac{x^n}{n!} = \frac{e^c x^{N+1}}{(N+1)!}$$ where c is between 0 and x.

If $x \in (0,R]$, then $0 < c < x \le R$.

So $1 = e^0 < e^c < e^x \le e^R$ ← e^x is str. inc.

$0 < c^{N+1} < x^{N+1} \le R^{N+1}$ ← so is x^{N+1} when $x > 0$.

Hence, $R = \frac{e^c x^{N+1}}{(N+1)!} > 0$ and $R = \frac{e^c x^{N+1}}{(N+1)!} < \frac{e^R R^{N+1}}{(N+1)!}$ ✓

$\int_{x^2}^{x}$

4. $$F(x) = \int_0^x \sqrt{1+t^7}\,dt - \int_0^{x^2} \sqrt{1+t^7}\,dt.$$ $0 < x^2 < x < 1$ when $0 < x < 1$

Let $G(u) = \int_0^u \sqrt{1+t^7}\,dt$. Then $F(x) = G(x) - \underbrace{G(x^2)}_{G(u) \text{ where } u = x^2}$

and the Fund Thm Calc and Chain rule yield that

$$F'(x) = G'(x) - G'(u)\frac{du}{dx} = \sqrt{1+x^7} - (2x)\sqrt{1+(x^2)^7},$$