1. (a) (5 points) Let $f : A \to B$, and let C, D be subsets of B. Prove

$$f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D).$$

Assume $x \in$ LHS. Show $x \in$ RHS. ✓

Assume $x \in$ RHS. Show $x \in$ LHS too.

Done.

$$f^{-1}(C \cap D) = \{x \in A : f(x) \in C \cap D\}$$

(b) (5 points) When E is a countable subset of $\mathbb{R}$, is the complement $\mathbb{R} \setminus E$ always uncountable? Explain why or why not.

Always uncountable.

If $\mathbb{R} \setminus E$ countable too, then

$$\mathbb{R} = E \cup (\mathbb{R} \setminus E)$$ ← Union of 2 countable sets! $\underbrace{\text{Countable!}}_{\mathbb{R}}$

$\varphi_1 : E \longleftrightarrow \mathbb{N}$

$\varphi_2 : \mathbb{R} \setminus E \longleftrightarrow \mathbb{N}$

$\varphi : \mathbb{R} \longleftrightarrow \mathbb{N}$

$$\varphi(N) = \begin{cases} \varphi_1(n) & : N = 2n \\ \varphi_2(n) & : N = 2n+1 \end{cases}$$

(c) (5 points) When E is an uncountable subset of $\mathbb{R}$, is the complement $\mathbb{R} \setminus E$ always countable? Explain why or why not.

Examples: $E =$ irrational #'s ← uncountable

$\mathbb{R} \setminus E =$ rationals ← countable

$E = [0, 1]$

$$\mathbb{R} \setminus E = (-\infty, 0) \cup (1, \infty)$$

Union of 2 uncountable sets is uncountable too.

(b) For each of the following scenarios, give an example satisfying the stated property. Formal proofs are not required, but some explanation may be useful.

(i) (5 points) A sequence $\{x_n\}$ converging to 0 which is not monotonic.

$$\frac{(-1)^n}{n}$$

(ii) (5 points) An unbounded sequence that has a convergent subsequence.

n

$$x_n = \begin{cases} 1 & n \text{ odd} \\ n & n \text{ even} \end{cases}$$

or $$\begin{cases} 1/n & n \text{ odd} \\ n & n \text{ even} \end{cases}$$

4. (a) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences of real numbers.

(i) (5 points) Prove that the sequence $\{x_n + y_n\}$ is bounded.

(ii) (5 points) Prove that

$$\limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

$\lim\sup_{n\to\infty} x_n = \lim_{N\to\infty} \sup\{x_n : n \geq N\}$

$x_m + y_m \leq \sup_{n\geq N} x_n + \sup_{n\geq N} y_n$

if $m \geq N$

So $\sup_{m\geq N}\{x_m + y_m\} \leq \not\leq \sup \leq$

Finally Lim left $\leq$ Lim right.

(b) (5 points) Let E denote the set of all real numbers in $(0,1)$ with decimal expansion involving only 1's and 2's:

$$E = \{x \in (0,1) : \forall j \in \mathbb{N}, \exists d_{-j} \in \{1,2\}, \text{ such that } x = 0.d_{-1}d_{-2}\dots\}.$$

Note that $0.2 \notin E$ but $0.222222\dots \in E$. Prove that $0.1111111\dots$ is a cluster point of E.

x_0 cluster pt means, given $\varepsilon > 0$, there is a pt $x \in E$ with $x_0 \neq x$ such that $|x_0 - x| < \varepsilon$.

Equiv to: There is a seq (x_n) in E with $x_n \neq x_0$ for all n converging to x_0.

$x_n = .1111\dots121\dots$ $\qquad$ $x = .1111\dots$

n-th decimal

$x_n - x = .0000\dots01000\dots$

$= 10^{-n}$

$\frac{1}{10^n} \to 0$ as $n \to \infty$. So can make $<$ any $\varepsilon > 0 \dots$

5. (a) (5 points) Suppose that for all $n \in \mathbb{N}$, $a_n > 0$, $b_n > 0$ and

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L > 0.$$

Prove that $\sum a_n$ converges if and only if $\sum b_n$ converges.

Compare tail ends of series