REAL ANALYSIS
FINAL ASSIGNMENT
DECEMBER 6, 2020

The following assignment consists of **7 problems** worth **15 points each**. Solutions should be written in complete sentences where appropriate.

You may freely use any knowledge of the functions $\sin x, \cos x, e^x$ and $\log x$ (the natural logarithm) that you retained from calculus. The assignment is open book, open Canvas notes and lectures, but **collaborating with other students or other sources on the internet is strictly prohibited**. By signing your name below, you attest to following these rules for the assignment. Evidence to the contrary will be treated as academic misconduct and will responded to according to MIT Institute Policy 10.2.

NAME:

1. Complete the following **negations**:

   (i) (5 points) Let $S \subset \mathbb{R}$. A function $f : S \to \mathbb{R}$ is **not continuous at** $c \in S$ if

   (ii) (5 points) Let $S \subset \mathbb{R}$. A function $f : S \to \mathbb{R}$ is **not uniformly continuous on** S if

(iii) (5 points) Let $S \subset \mathbb{R}$. A sequence of functions $f_n : S \to \mathbb{R}$ **does not converge uniformly** to $f : S \to \mathbb{R}$ if

2. (a) Give explicit examples satisfying the stated conditions. Explanation, but not a formal proof, for each case is required.

   (i) (5 points) A continuous function on $(0, 1)$ with neither a global maximum nor minimum.

   (ii) (5 points) A function on $[0, 1]$ with an absolute minimum at 0, absolute maximum at 1 and such that there exists $y \in (f(0), f(1))$ not in the range of f.

(b) (5 points) Let $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ be functions continuous at $c \in S$. Prove that the product $fg : S \to \mathbb{R}$ is continuous at c.

3. We say a subset $K \subset \mathbb{R}$ is **compact** if for every sequence $\{x_n\}_n$ of elements of K, there exists a subsequence $\{x_{n_k}\}_k$ and $x \in K$ such that $\lim_{k\to\infty} x_{n_k} = x$ (every sequence has a subsequence converging in K).

   (a) (7 points) Let $a < b$. Prove that $[a, b]$ is compact. *Hint*: A theorem named after two men, one with an Italian surname and the other with a German surname, might be useful.

(b) (8 points) Let $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a continuous function. Prove that

$$f([a, b]) := \{f(x) : x \in [a, b]\}$$

is compact.

4. (a) (i) (5 points) Let $a < b$. Prove that if $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$ then f is continuous at c.

(ii) (5 points) Give an explicit example of a function showing the converse of part (i) is false.

(b) (5 points) Let

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Use the definition of the derivative to prove that f is differentiable at 0.

5. (a) (7 points) Let $R > 0$. Prove that for all $x \in [-R, R]$,

$$e^x - \sum_{j=0}^{n} \frac{x^j}{j!} \; \leq \frac{e^R R^{n+1}}{(n+1)!}.$$

(b) (8 points) Suppose $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable. Prove that for all $c, x \in \mathbb{R}$ with $c < x$,

$$f(x) = f(c) + f'(c)(x - c) + \int_c^x f''(t)(x - t)dt.$$

Hint: Compute the integral using a theorem which ends in "parts."

6. (a) (5 points) Compute

$$\int_0^1 x \log x dx := \lim_{a \to 0^+} \int_a^1 x \log x dx.$$

You may use without proof the following version of **L'Hôpital's rule**: if $f, g : (0,1) \to \mathbb{R}$, $g(x) \neq 0$ for all x, $f(x) \to -\infty$, $g(x) \to \infty$ as $x \to 0^+$, and $L = \lim_{x \to 0^+} f'(x)/g'(x)$ exists, then $\lim_{x \to 0^+} f(x)/g(x) = L$.

(b) (5 points) Prove

$$\lim_{n \to \infty} \int_0^1 x^n \sin(x) dx = 0.$$

(c) (5 points) Let $f : [-\pi, \pi] \to \mathbb{R}$ be a continuously differentiable function. Prove that

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \sin(nx) f(x) dx = 0.$$

Hint: $\sin(nx) = \left[-\frac{1}{n}\cos(nx)\right]'$

7. (a) (5 points) Give an explicit example of a sequence of continuous functions on $(0,1)$ that converges pointwise to a continuous function on $(0,1)$ but the convergence is not uniform. Explanation, but not a formal proof, is required. Feel free to use pictures.

(b) Suppose $f:\mathbb{R}\to\mathbb{R}$ is a differentiable function and there exists $L>0$ such that

$$|f'(c)| \leq L, \quad \forall c \in \mathbb{R}.$$

(i) (5 points) Prove that $f:\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous on $\mathbb{R}$.

(ii) (5 points) Define a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) := f\left(x + \frac{1}{n}\right).$$

Prove that $\{f_n\}_n$ converges to f uniformly on $\mathbb{R}$.