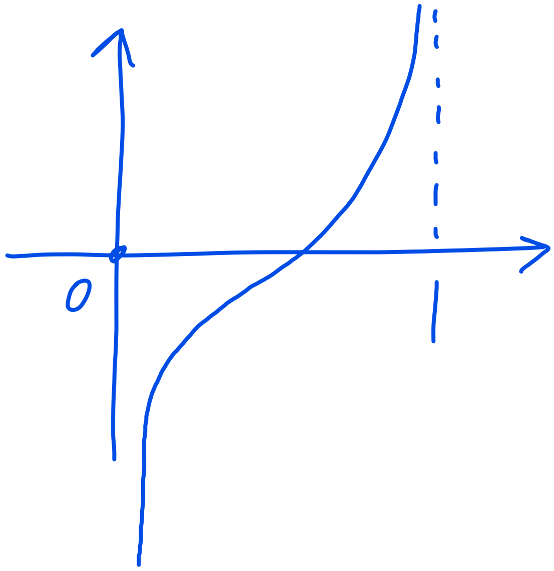
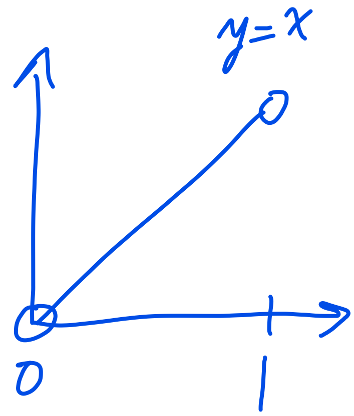
2. (a) Give explicit examples satisfying the stated conditions. Explanation, but not a formal proof, for each case is required.

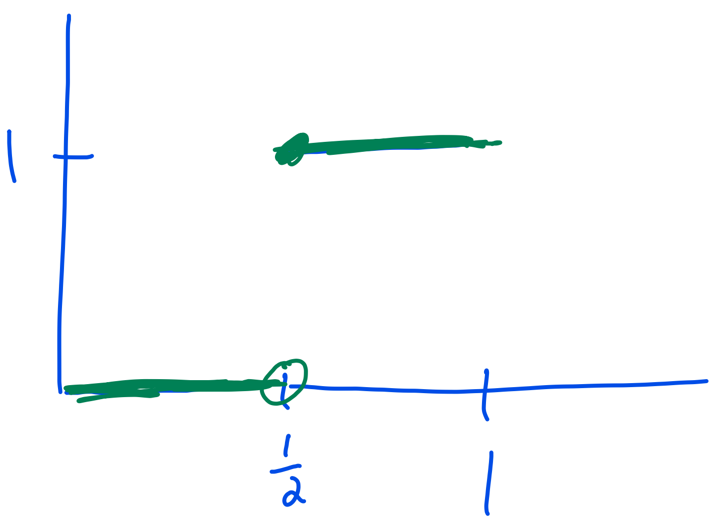
   (i) (5 points) A continuous function on $(0, 1)$ with neither a global maximum nor minimum.





   (ii) (5 points) A function on $[0, 1]$ with an absolute minimum at 0, absolute maximum at 1 and such that there exists $y \in (f(0), f(1))$ not in the range of f.

Not continous. IVT ↯



(b) (5 points) Let $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ be functions continuous at $c \in S$. Prove that the product $fg : S \to \mathbb{R}$ is continuous at c.

$$f(x)g(x) - f(c)g(c)$$

$$= f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)$$

$$= \left(f(x) - f(c)\right)\underbrace{g(x)} + f(c)\left(g(x) - g(c)\right)$$

Need to know $|g(x)| < M$ on interval about c. ✓

$$\frac{\varepsilon}{2} \cdot M + |f(c)| \cdot \frac{\varepsilon}{2}$$

$$\tilde{\varepsilon} = \frac{\varepsilon}{\operatorname{Max}(M, |f(c)|)}$$

3. We say a subset $K \subset \mathbb{R}$ is **compact** if for every sequence $\{x_n\}_n$ of elements of K, there exists a subsequence $\{x_{n_k}\}_k$ and $x \in K$ such that $\lim_{k\to\infty} x_{n_k} = x$ (every sequence has a subsequence converging in K).

   (a) (7 points) Let $a < b$. Prove that $[a, b]$ is compact. *Hint*: A theorem named after two men, one with an Italian surname and the other with a German surname, might be useful.

Bolzano-Weierstraß!

4. (a) (i) (5 points) Let $a < b$. Prove that if $f : (a,b) \to \mathbb{R}$ is differentiable at $c \in (a,b)$ then f is continuous at c.

$$\frac{f(x) - f(c)}{x - c} - f'(c) = E(x)$$

defines $E(x)$, $x \neq c$.

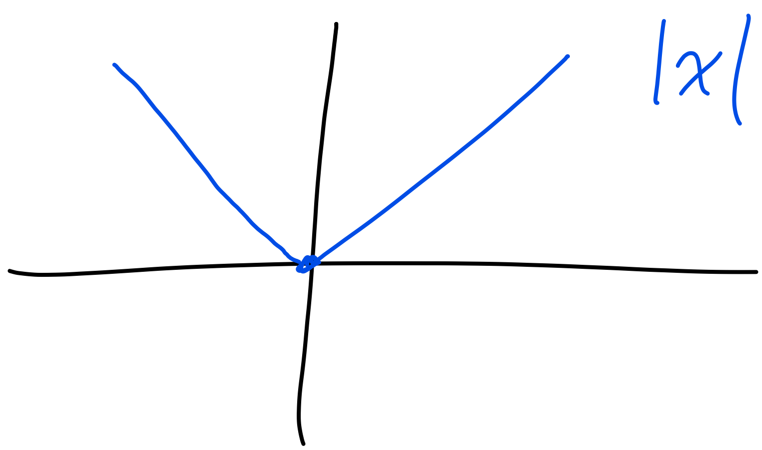
$E(x) \to 0$ as $x \to c$.

$$f(x) - f(c) = f'(c)(x-c) + E(x)(x-c)$$

$$\longrightarrow f'(c) \cdot 0 + 0 \cdot 0 = 0$$

as $x \to c$.

(ii) (5 points) Give an explicit example of a function showing the converse of part (i) is false.



(b) (5 points) Let

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Use the definition of the derivative to prove that f is differentiable at 0.

$$DQ = \frac{x^2 \sin\frac{1}{x} - 0}{x - 0}$$

$$= x \underbrace{\sin\frac{1}{x}}$$

$$\left|\sin\frac{1}{x}\right| \leq M = 1$$

$\varepsilon > 0.$ $\qquad \left|x \sin\frac{1}{x} - 0\right| \leq |x| \cdot 1$

$\delta = \varepsilon.$ $\quad$ [or $\delta = \frac{\varepsilon}{M}$.]

5. (a) (7 points) Let $R > 0$. Prove that for all $x \in [-R, R]$,

$$0 \overset{?}{\le} \underbrace{e^x - \sum_{j=0}^{n} \frac{x^j}{j!}} \le \frac{e^R R^{n+1}}{(n+1)!}.$$

? Yes, if N odd, $x<0$
No, if N even, $x<0$
$=0$ at $x=0$.

$$\frac{e^c x^{N+1}}{(N+1)!}$$

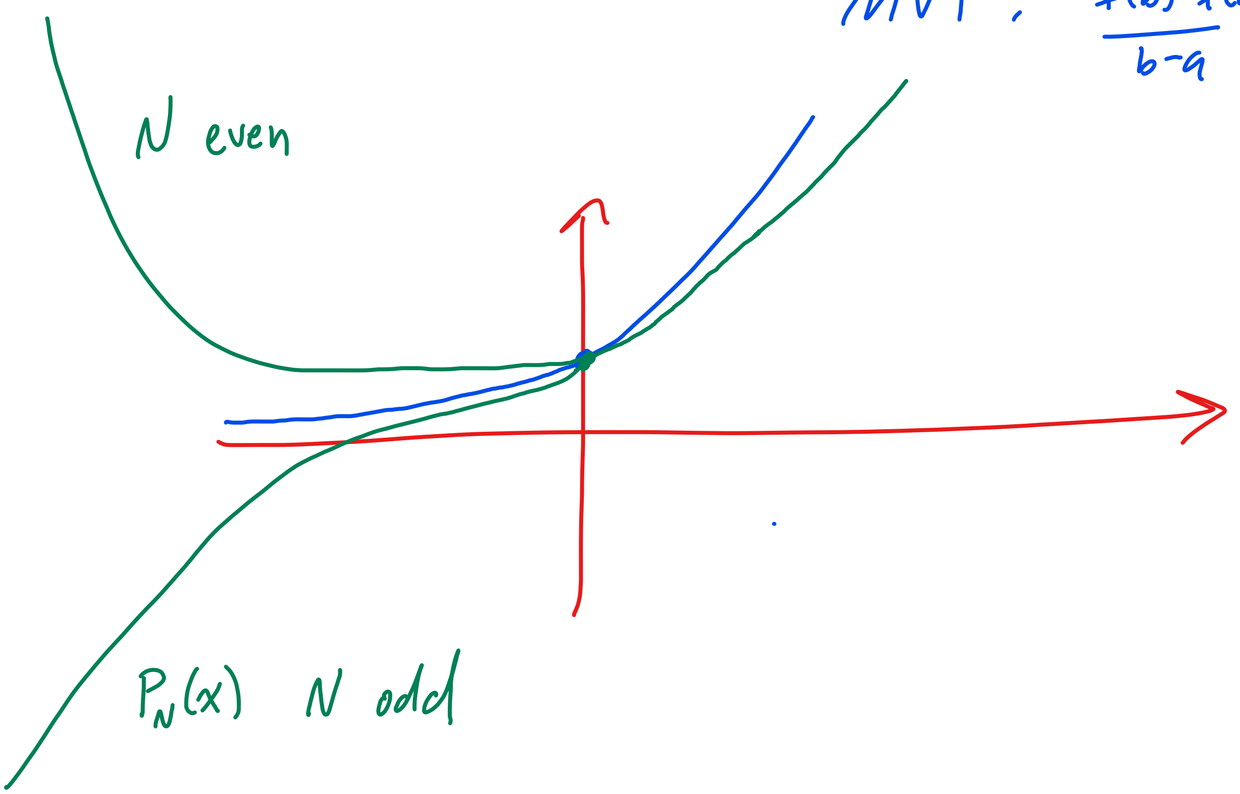
$$\frac{f^{(N+1)}(c)}{(N+1)!}(x-x_0)^{N+1}$$

$-R \le x < c < R$

e^x strictly increasing because $(e^x)' = e^x > 0$.

$0 < e^c < e^R$

MVT: $\frac{f(b)-f(a)}{b-a} = \underbrace{f'(c)}_{>0}$



6. (a) (5 points) Compute

$$\int_0^1 x\log x\,dx := \lim_{a\to 0^+}\int_a^1 x\log x\,dx.$$

Int. by parts; $\begin{cases} u = \ln x \\ dv = x\,dx \end{cases}$

You may use without proof the following version of **L'Hôpital's rule**: if $f, g : (0,1) \to \mathbb{R}$, $g(x) \neq 0$ for all x, $f(x) \to -\infty$, $g(x) \to \infty$ as $x \to 0^+$, and $L = \lim_{x\to 0^+} f'(x)/g'(x)$ exists, then $\lim_{x\to 0^+} f(x)/g(x) = L$.

$$\underset{0\cdot(-\infty)}{x\ln x} = \frac{\ln x}{\left(\frac{1}{x}\right)} \xrightarrow{\frac{(-\infty)}{\infty}} \lim_{x\to 0+} \frac{\left(\frac{1}{x}\right)}{\left(\frac{-1}{x^2}\right)} = 0$$

$f(x) = \begin{cases} x\ln x & x>0 \\ 0 & x=0 \end{cases}$ is cont. on $[0,1]$. So $\int_0^1$ exists.

$$\left|\int_0^1 - \int_a^1\right| = \left|\int_0^a\right| \le \underbrace{\left(\text{Max}|\text{fcn}|\right)}_{\substack{\text{assumes its Max.}\\ \text{because cont on } [0,1].}} \underbrace{\text{Length}}_{a}$$

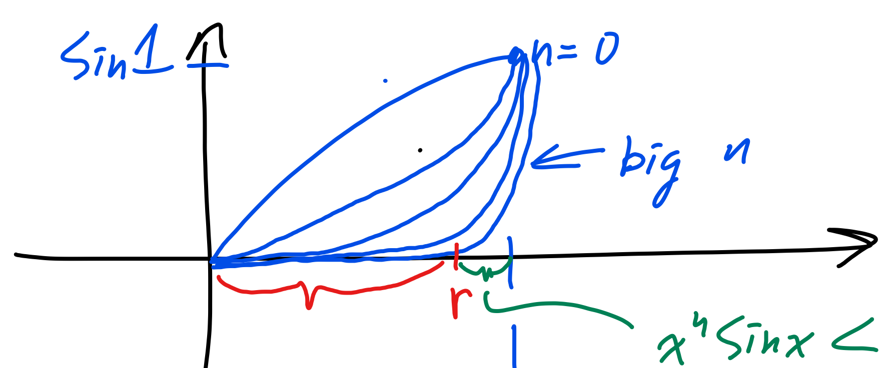
(b) (5 points) Prove

$$\lim_{n\to\infty}\int_0^1 \underbrace{x^n\sin(x)}\,dx = 0.$$

$\to 0$ pointwise

($x=1$: $=\sin 1$)

Know $x^n \to 0$ as $n\to\infty$ when $0\le x<1$.



$x^n\sin x < \sin 1 < 1$

$x^n \sin x < r^n$

$$\int_r^1 x^n\sin x\,dx < 1\cdot(1-r)$$

Choose r so that $(1-r)<\frac{\varepsilon}{2}$. Next, choose n so that $r^n<\frac{\varepsilon}{2}$.

(c) (5 points) Let $f : [-\pi, \pi] \to \mathbb{R}$ be a continuously differentiable function. Prove that

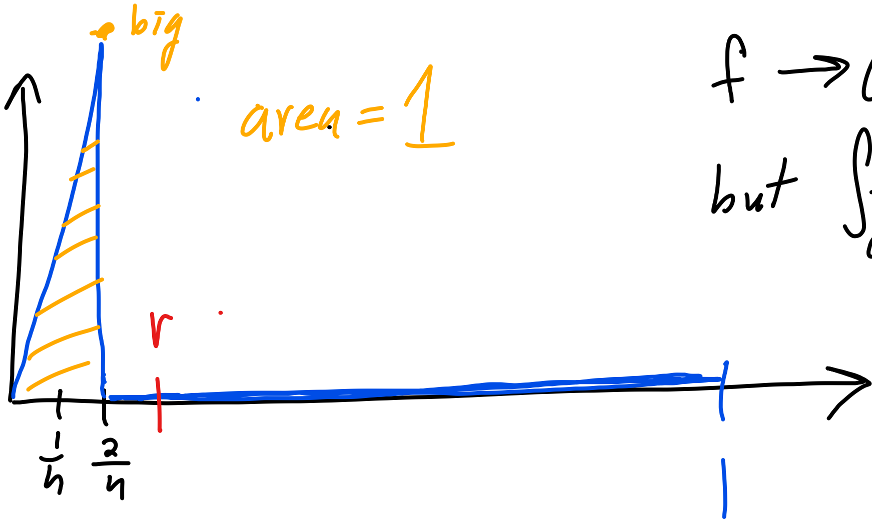
$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \sin(nx)\underbrace{f(x)}_{u}dx = 0.$$

$dv = \sin(nx)\,dx$

Hint: $\sin(nx) = \left[-\frac{1}{n}\cos(nx)\right]'$

Integrate by parts!

7. (a) (5 points) Give an explicit example of a sequence of continuous functions on $(0,1)$ that converges pointwise to a continuous function on $(0,1)$ but the convergence is not uniform. Explanation, but not a formal proof, is required. Feel free to use pictures.



$f \to 0$ pointwise.

but $\int_0^1 f \not\to 0$.

(b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function and there exists $L > 0$ such that

$$|f'(c)| \leq L, \quad \forall c \in \mathbb{R}.$$

(i) (5 points) Prove that $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous on $\mathbb{R}$.

MVT

$$\left| \frac{f(x_2) - f(x_1)}{x_2 - x_1} \right| = |f'(c)| \leq M$$

$$|f(x_2) - f(x_1)| \leq M \, |x_2 - x_1|$$

(ii) (5 points) Define a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) := f\left(x + \frac{1}{n}\right).$$

f Lipschitz

Prove that $\{f_n\}_n$ converges to f uniformly on $\mathbb{R}$.

otherwise, can't.

$$\left| f(x+\tfrac{1}{n}) - f(x) \right|$$

$$\le \underbrace{M \left| (x+\tfrac{1}{n}) - x \right|}_{\frac{M}{n}} \longrightarrow 0 \text{ as } n \to \infty$$

(Archimedes princ.)

Given $\varepsilon > 0$. Get N such estimate works for all $x \in \mathbb{R}$ when $n \ge N$.

Off hrs. Wed, Thur 2-3 pm.