MATH 366

Exam 1

1. (20 pts) The second order ODE

$$(*) \qquad -x\frac{d^2y}{dx^2} = \frac{v_R}{v_D}\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

was derived and studied in Lessons 1 and 2 to model the path a dog, starting at the point $(a,0)$ on the negative x-axis, follows as it chases a rabbit that starts at the origin and runs straight up the y-axis. The dog and rabbit are assumed to run at constant speeds v_D and v_R, respectively, and the dog is assumed to run in the direction of the position of the rabbit at any instant. Hence

$$\frac{dy}{dx}(a) = 0, \qquad \text{and} \qquad y(a) = 0$$

are initial conditions for the problem.

a) Let $p = \frac{dy}{dx}$ and write (*) as a first order ODE in p. Solve for p to get

$$p = \sinh\left(-\frac{v_R}{v_D}\ln|x| + C\right)$$

and use the initial conditions to determine C.
(Note that $|x| = -x$ when x is negative, as is the case in this problem, and that a is also negative.)

b) Assume $a = -1$ and $v_D = v_R$ and use the fact that $\sinh u = \frac{1}{2}(e^u - e^{-u})$ and properties of exponentials and log to get an expression of the form $c_1(-x)^{q_1} + c_2(-x)^{q_2}$ for p (and determine those constants c_1, c_2, q_1, and q_2). Now replace p by $\frac{dy}{dx}$, integrate, and use the initial conditions to find the pursuit path $(x, y(x))$ in this case. Does the dog ever catch the rabbit?

c) Repeat part (b) using $a = -1$ and $v_D = 2v_R$. Find the point where the dog catches the rabbit in this case.

2. (20 pts) If $y = h_1(x)$ and $y = h_2(x)$ are two continuously differentiable solutions to

$$\frac{dy}{dx} = f(x,y)$$

where $f(x,y)$ is a continuous function on the whole xy-plane, explain why the graphs of the two solutions cannot cross, i.e., why there can be no point x_0 where h_1 and h_2 agree, but their derivatives do not agree.

Give an additional condition on $f(x,y)$ that would guarantee that the graphs of the two solutions cannot pass through a point (x_0, y_0) with the same tangent slope without being the same solution.

3. (20 pts) Convert the first order ODE

$$\frac{dy}{dx} = \sin(xy)$$

into a new ODE

$$\frac{du}{dx} = F(x, u)$$

by making the change of variables $u = xy$. Find the function $F(x, u)$ in the new equation explicitly. DO NOT TRY TO SOLVE THE NEW EQUATION.

4. (20 pts)

a) Solve the initial value problem

$$2y' + y = e^{-5x} \quad \text{with } y(0) = 1.$$

b) If $g(x)$ is a continuous function, find a formula for the solution to

$$2y' + y = g(x) \quad \text{with } y(0) = 0$$

of the form $y(x) = \int_0^x K(x,t)g(t)\,dt$. Find the "kernel" $K(x,t)$ and explain why it makes sense to say that the present value of y depends on *all* the values of $g(x)$ since $x = 0$. Note that your formula for $y(x)$ is particularly useful if $g(x)$ is given by data because straightforward numerical methods can be used to compute the integral.

5. (20 pts) Show that the integrating factor $u(x) = x^{-3}$ turns the ODE

$$(2x - y^2) + (xy)\frac{dy}{dx} = 0$$

into an *exact* first order ODE. Use it to find the *general solution* to the original equation. Find the solution satisfying $y(2) = 4$ and write the answer in the form $y =$ (a function of x), if you can.