MA 366 Exam 2 sol^{n}s and grading key

1. Homogeneous solution: $r^2+4r+4=0$

$(r+2)^2=0 \quad r=-2,-2$

$$y = \underline{c_1 e^{-2x} + c_2 x e^{-2x}} \longleftarrow 3 \text{ pts}$$

Particular sol^n: $y_p = A\cos 3x + B\sin 3x$

$y_p' = -3A\sin 3x + 3B\cos 3x$

$y_p'' = -9A\cos 3x - 9B\sin 3x$

Plug into ODE:

$$(-9A\cos 3x - 9B\sin 3x) + 4(-3A\sin 3x + 3B\cos 3x) + 4(A\cos 3x + B\sin 3x)$$

$$= \left[\underbrace{-5A+12B}_{=0}\right]\cos 3x + \left[\underbrace{-12A-5B}_{=1}\right]\sin 3x = \sin 3x$$

$$\begin{bmatrix} -5 & 12 \\ -12 & -5 \end{bmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$A = \frac{\det\begin{bmatrix} 0 & 12 \\ 1 & -5 \end{bmatrix}}{\det\begin{bmatrix} -5 & 12 \\ -12 & -5 \end{bmatrix}} = \frac{-12}{169}$$

$$B = \frac{\det\begin{bmatrix} -5 & 0 \\ -12 & 1 \end{bmatrix}}{\det\begin{bmatrix} -5 & 12 \\ -12 & -5 \end{bmatrix}} = \frac{-5}{169}$$

So $\underline{y_p = \frac{-12}{169}\cos 3x - \frac{5}{169}\sin 3x}$ ← 5pts

and the general solution is

$$\underline{y = c_1 e^{-2x} + c_2 x e^{-2x} + \left(\frac{-12}{169}\cos 3x - \frac{5}{169}\sin 3x\right)}$$ ← 2pts

i.e., (gen^l sol^n to homog) + y_p

2.a) $\left.\begin{aligned} y &= c_1 x^2 + c_2 x^3 \\ y' &= 2c_1 x + 3c_2 x^2 \end{aligned}\right\}$

$$c_1 = \frac{\det\begin{bmatrix} y & x^3 \\ y' & 3x^2 \end{bmatrix}}{\det\begin{bmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{bmatrix}} = \underline{\frac{3}{x^2}y - \frac{1}{x}y'}$$

$$c_2 = \frac{\det\begin{bmatrix} x^2 & y \\ 2x & y' \end{bmatrix}}{\det\begin{bmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{bmatrix}} = \underline{\frac{1}{x^2}y' - \frac{2}{x^3}y}$$

$$y'' = 2c_1 + 6c_2 x$$

$$y'' = 2\left(\frac{3}{x^2}y - \frac{1}{x}y'\right) + 6\left(\frac{1}{x^2}y' - \frac{2}{x^3}y\right)x$$

$$\boxed{x^2 y'' - 4xy' + 6y = 0}$$ ← 10 pts

b) If $\begin{cases} c_1 x^2 + c_3 x^3 \equiv 0, \\ c_1 2x + 3c_3 x^2 \equiv 0 \end{cases}$ then the derivative too

Plug in $x=1$:
$$c_1 + c_3 = 0$$
$$2c_1 + 3c_3 = 0$$

Note: $W(1) \neq 0$ $\Rightarrow y_1, y_2$ indep. on $\mathbb{R}$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}}_{\det = 3-2 = 1 \neq 0} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \vec{0}$$

$c_1 = 0$ and $c_2 = 0$ is the only possible choice,

So x^2 and x^3 are linearly independent

5 points for correct answer

c) $$W[x^2, x^3] = \det \begin{bmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{bmatrix}$$

$$= 3x^4 - 2x^4 = x^4 \leftarrow 3\text{pts}$$

x^4 vanishes at $x=0$. This does not violate Abel's result because the ODE is $y'' - \frac{4}{x} y' + \frac{6}{x^2} y = 0$, i.e., $y'' + P(x) y' + Q(x) y = 0$ where P and Q are singular at $x=0$. Abel's result requires

P and Q to be continuous on the interval under consideration. ← 2pts

3. a) Suppose $c_1 y_1 + c_2 y_2 \equiv 0$

<u>For $x<0$:</u> $c_1(x^3) + c_2(-x^3) \equiv 0$

Divide by x^3 (or set $x=-1$) to get

$$c_1 - c_2 = 0$$

<u>For $x>0$:</u> $c_1 x^3 + c_2 x^3 \equiv 0$

Divide by x^3 (or set $x=1$) to get

$$c_1 + c_2 = 0$$

The only sol^n to $\begin{cases} c_1 - c_2 = 0 \\ c_1 + c_2 = 0 \end{cases}$

is $c_1 = 0$, $c_2 = 0$, so y_1 and y_2 are linearly independent. 5 pts

b) For $x<0$, $W[y_1, y_2] = \det \begin{bmatrix} x^3 & -x^3 \\ 3x^2 & -3x^2 \end{bmatrix} \equiv 0.$

For $x \geq 0$, $W = \det \begin{bmatrix} x^3 & x^3 \\ 3x^2 & 3x^2 \end{bmatrix} \equiv 0.$

So $W \equiv 0$ on $\mathbb{R}$. 5 pts

4. $y = x^r$

$y' = r x^{r-1}$

$y'' = r(r-1) x^{r-2}$

Plug in ODE: $x^2 \left[r(r-1) x^{r-2} \right] - 3x \left[r x^{r-1} \right] + 4 x^r$

$$= \left[r(r-1) - 3r + 4 \right] x^r \overset{\text{want}}{=} 0$$

$$= (r^2 - 4r + 4) x^r$$

$$= \underbrace{(r-2)^2 x^r}_{5 \text{ pts}} \equiv 0 \text{ only if } r = 2.$$

Standard form of homogeneous eqn:

$$y'' + \underbrace{\left(\frac{-3}{x}\right)}_{P(x)} y' + \left(\frac{4}{x^2}\right) y = 0$$

$y_2 = u y_1$ where $u' = \frac{1}{y_1^2} e^{-\int P(x)\, dx}$

$$= \frac{1}{(x^2)^2} e^{-\int -\frac{3}{x} dx}$$

$$= \frac{1}{x^4} e^{3 \ln x} = \frac{1}{x^4} \underbrace{e^{\ln x^3}}_{x^3}$$

$$u' = \frac{1}{x}$$

So $u = \int \frac{1}{x} dx = \ln x$

and we get $y_2 = u y_1 = \underline{(\ln x) x^2}$ ← 5 pts

Next $y_p = u_1 y_1 + u_2 y_2$ where

$$u_1' = \frac{-y_2 F}{W} \quad \text{and} \quad u_2' = \frac{y_1 F}{W}$$

where $y'' + \left(\frac{-3}{x}\right) y' + \left(\frac{4}{x^2}\right) y = \underbrace{\ln x}_{F(x)}$

from standard form and

$$W = \det \begin{bmatrix} x^2 & x^2 \ln x \\ 2x & 2x \ln x + x \end{bmatrix} = x^3$$

So $u_1' = \frac{-(x^2 \ln x)(\ln x)}{x^3}$ $\quad u_2' = \frac{(x^2)(\ln x)}{x^3}$

$$u_1' = -\frac{1}{x}(\ln x)^2 \qquad u_2' = \frac{1}{x}\ln x$$

$$u_1 = -\int (\underbrace{\ln x}_{v})^2 \left(\underbrace{\frac{1}{x}dx}_{dv}\right) \qquad u_2 = \int (\ln x)\left(\frac{1}{x}dx\right)$$

$$u_1 = -\frac{1}{3}(\ln x)^3 \qquad u_2 = \frac{1}{2}(\ln x)^2$$

So $y_p = -\frac{1}{3}(\ln x)^3(x^2) + \frac{1}{2}(\ln x)^2(x^2\ln x)$

$$= \frac{1}{6}x^2(\ln x)^3 \leftarrow 5 \text{ pts}$$

General solution:

$$y = \underline{c_1x^2 + c_2x^2\ln x + \frac{1}{6}x^2(\ln x)^3}$$

5 pts

5. $r^4 + 2r^2 + 1 = (r^2+1)^2 = 0$

Roots: $r = i, i, -i, -i$

General solution to $y^{(4)} + 2y'' + y = 0$ is

$$y = \underline{c_1\cos x + c_2\sin x + c_3x\cos x + c_4x\sin x}$$

10 pts

Correct form of y_p:

$$y_p = x^2\left[(A_1 x + A_0)\cos x + (B_1 x + B_0)\sin x\right]$$

10 pts

6.

$$\vec{x}\,' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\vec{x} + \begin{pmatrix} 5 \\ 10 \end{pmatrix} e^{2t}$$

$$\det(A - rI) = \det\begin{bmatrix} -2-r & 1 \\ 1 & -2-r \end{bmatrix} = r^2 + 4r + 3$$

$$= (r+1)(r+3) = 0 \qquad r = -1, -3$$

For $r=-1$:

$$\left[\begin{array}{cc|c} -2-(-1) & 1 & 0 \\ 1 & -2-(-1) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector. $\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$ ⟸ 5pts

For $r=-3$:

$$\left[\begin{array}{cc|c} -2-(-3) & 1 & 0 \\ 1 & -2-(-3) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector $\vec{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t}$ 5pts

Finally, get $\vec{x}_p = \vec{b} e^{2t}$ using method of undet coeff:

$$\vec{x}_p{}' = \mathbb{A} \vec{x}_p + \begin{pmatrix} 5 \\ 10 \end{pmatrix} e^{2t}$$

$$2\vec{b} e^{2t} = \mathbb{A} \vec{b} e^{2t} + \begin{pmatrix} 5 \\ 10 \end{pmatrix} e^{2t}$$

$$\begin{pmatrix} -5 \\ -10 \end{pmatrix} = \mathbb{A} \vec{b} - 2\vec{b} = \begin{pmatrix} -2b_1 + b_2 \\ b_1 - 2b_2 \end{pmatrix} - \begin{pmatrix} 2b_1 \\ 2b_2 \end{pmatrix}$$

$$\begin{pmatrix} -5 \\ -10 \end{pmatrix} = \begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Cramer's rule: $b_1 = \dfrac{\det \begin{bmatrix} -5 & 1 \\ -10 & -4 \end{bmatrix}}{\det \begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix}} = \dfrac{30}{15} = 2$

$$b_2 = \frac{\det \begin{bmatrix} -4 & -5 \\ 1 & -10 \end{bmatrix}}{\det \begin{bmatrix} -4 & 1 \\ 1 & -4 \end{bmatrix}} = \frac{45}{15} = 3$$

$$\vec{x}_p = \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{2t}$$

5pts

and the general solution is

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^{2t}$$

5 pts

or

$$\begin{cases} x_1 = c_1 e^{-t} + c_2 e^{-3t} + 2e^{2t} \\ x_2 = c_1 e^{-t} - c_2 e^{-3t} + 3e^{2t} \end{cases}$$