MATH 366

Exam 2

1. (20 pts) Find the solution to

$$y'' + 4y' + 4y = e^{-t}$$

satisfying $y(0) = 0$ and $y'(0) = 0$.

2. (20 pts) Find a complex particular solution of the equation

$$y'' + 2y' + y = e^{i2t}$$

of the form Ae^{i2t} where A is a complex constant.

3. (20 pts) A real 2×2 matrix $\mathbb{A}$ has a complex eigenvalue $2 + 3i$ with corresponding complex eigenvector $\vec{a} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$. Find a *real* general solution to $\vec{x}\,' = \mathbb{A}\vec{x}$.

4. (20 pts) The matrix $\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ has eigenvalues 1 and -1 with corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, respectively. Find the solution to $\vec{x}\,' = \mathbb{A}\vec{x}$ satisfying $\vec{x}(0) = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

5. (20 pts) What is the type (sprial, center, node, saddle) and stability (stable, asymptotically stable, unstable) of the critical point at the origin of the linear system

$$\vec{x}\,' = \begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix} \vec{x}\,?$$

Explain.

Exam 2 Solutions

1. $r^2 + 4r + 4 = 0$

$(r+2)^2 = 0 \qquad r = -2, -2$

Homog. sol^n; $c_1 e^{-2t} + c_2 t e^{-2t}$

Part. sol^n; $y_p = Ae^{-t}$

$y_p' = -Ae^{-t}$

$y_p'' = Ae^{-t}$

want

Need $(Ae^{-t}) + 4(-Ae^{-t}) + 4Ae^{-t} = e^{-t}$

$Ae^{-t} = e^{-t}$

$\boxed{A = 1}$

Gen^l sol^n; $y = c_1 e^{-2t} + c_2 t e^{-2t} + e^{-t}$

$y' = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t} - e^{-t}$

want

$y(0) = c_1 + 0 + 1 = 0 \qquad \boxed{c_1 = -1}$

$y'(0) = -2c_1 + c_2 - 1 = 0 \qquad c_2 = 1 + 2c_1 = 1 - 2 = -1$

$\boxed{c_2 = -1}$

Ans: $y = \underline{\underline{-e^{-2t} - te^{-2t} + e^{-t}}}$

2. $y_p = Ae^{i2t}$

$y_p' = i2Ae^{i2t}$

$y_p'' = -4Ae^{i2t}$

Need $(-4Ae^{i2t}) + 2(i2Ae^{i2t}) + Ae^{i2t} \overset{\text{want}}{=} e^{i2t}$

$$[-3+4i]Ae^{i2t} = e^{i2t}$$

$$[-3+4i]A = 1$$

$$A = \frac{1}{-3+4i} \cdot \frac{-3-4i}{-3-4i} = \frac{-3-4i}{9+16} = -\frac{3}{25} - \frac{4}{25}i$$

Complex particular sol^n: $y_p = \underline{\underline{\left(-\frac{3}{25} - \frac{4}{25}i\right)e^{i2t}}}$

$$= \left(-\frac{3}{25} - \frac{4}{25}i\right)(\cos 2t + i\sin 2t)$$

$$= \left(-\frac{3}{25}\cos 2t + \frac{4}{25}\sin 2t\right) + i\underbrace{\left(-\frac{4}{25}\cos 2t - \frac{3}{25}\sin 2t\right)}_{\substack{\text{particular sol}^n \\ \text{to } y''+2y'+y=\sin 2t}}$$

$$L[u+iv] = \underbrace{L[u]}_{\cos 2t} + i\underbrace{L[v]}_{\sin 2t} = e^{i2t}$$

3. Complex sol^n: $\begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{(2+3i)t}$

$$= \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right] \left(e^{2t}\cos 3t + i e^{2t} \sin 3t \right)$$

$$= \underbrace{\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}\cos 3t - \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^{2t} \sin 3t \right\}}_{\vec{x}_1} + i \underbrace{\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}\sin 3t + \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^{2t} \cos 3t \right\}}_{\vec{x}_2}$$

Gen^l sol^n: $\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$

$$= c_1 e^{2t} \begin{pmatrix} \cos 3t \\ -2\sin 3t \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} \sin 3t \\ 2\cos 3t \end{pmatrix}$$

4. Gen^l sol^n: $\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$

Need $\vec{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & 3 & 5 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 2 & 2 \end{array}\right] \begin{array}{l} \leftarrow c_1 + c_2 = 3 \\ \leftarrow 2c_2 = 2 \quad \boxed{c_2 = 1} \end{array}$$

$c_1 = 3 - c_2 = 3 - 1 = 2 \quad \boxed{c_1 = 2}$

Ans: $\vec{x} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} = \begin{pmatrix} 2e^{t} + e^{-t} \\ 2e^{t} + 3e^{-t} \end{pmatrix}$

5. $\det \begin{bmatrix} -3-r & -1 \\ 2 & -1-r \end{bmatrix} = (-3-r)(-1-r) + 2$

$= r^2 + 4r + 5 = 0$

$r = \dfrac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$

r complex: $\underline{\underline{\operatorname{Re} r = -2 < 0}}$

Type: Spiral (in)

Stability: Asymptotically stable

Explain: Sol^n is $\vec{x} = c_1 e^{-2t} \begin{pmatrix} \text{periodic} \\ \text{fcns} \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} \text{periodic} \\ \text{fcns} \end{pmatrix}$,

which spirals <u>in</u> as $e^{-2t} \to 0$ while $t \to \infty$.